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UNIVERSITY OF ALBERTA

# Groups With Permutable Subgroups And Infinite Metabelian Groups

BY

**Pan Soo Kim**

(C)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL 1991

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ISBN 0-315-70058-0

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**Date:** Aug 15 1991

**To my parents**

**for the life**

**long support**

## ABSTRACT

The first part of this thesis is the study of the following type. Let $u(x_1,\ldots,x_n) = x_{11}\cdots x_{1m}$ be a word in the alphabet $x_1,\ldots,x_n$ such that $x_{1i} \neq x_{1i+1}$ for all $i = 1,\ldots,m-1$. If $(H_1,\ldots H_n)$ is an $n$-tuple of subgroups of a group $G$ then denote by $u(H_1,\ldots,H_n)$ the set $\{u(h_1,\ldots,h_n) \mid h_i \in H_i\}$. If $\sigma \in S_n$ then denote by $u_\sigma(H_1,\ldots,H_n)$ the set $u(H_{\sigma(1)},\ldots,H_{\sigma(n)})$. We study groups $G$ with the property that for each $n$-tuple $(H_1,\ldots,H_n)$ of subgroups of $G$, there is some $\sigma \in S_n$, $\sigma \neq 1$ such that $u(H_1,\ldots,H_n) = u_\sigma(H_1,\ldots H_n)$. In chapter 2, we show that if $G$ is finitely generated soluble groups, then $G$ has this property for some word $u$ if and only if $G$ is nilpotent-by-finite. We also look at some specific words $u$ and study the properties of the associated groups.

For the middle parts, we consider two kinds of group properties and will see easily the one includes the other. Suppose $G$ is an infinite group with the property that whenever $X_1, X_2, X_3, X_4$ are infinite subsets of $G$ there exist $x_i \in X_i$ $(i = 1,2,3,4)$ such that

(i) $\langle x_1, x_2, x_3, x_4\rangle$ *is metabelian* or,

(ii) $[\,[x_1,x_2],[x_3,x_4]\,] = 1$

We show any infinite group satisfying (i) is metabelian and any locally soluble group satisfying (ii) is also metabelian.

For the final part, a finitely generated infinite group $G$ is proved to be quasi-Hamiltonian if any two infinite sets of subgroups of $G$ contain pair H, K that permute. This is not true for arbitrary infinite groups, a counter example is provided.

## ACKNOWLEDGEMENTS

I wish to express my sincere and warmest thanks to my supervisor, Dr. A.H. Rhemtulla for his invaluable guidance and continuous encouragement throughout my studies at the University of Alberta. Without him , this work would not have been possible.

I would also like to thank Dr. M. Shirvani and Dr. M. Maj for their care and support of this thesis. Moreover I am indebted to them for all their comments and discussions during my years of study.

Special thanks go to Dr. S.K. Sehgal and Dr. B.N. Allision for the reading of this thesis.

I extend my thanks to Professor A.C. Kim who has continually supported and watched over me. I also feel deep gratitude to him for giving me this opportunity.

Lastly, all my love and gratitude to my wife Kelley, who typed this thesis and for standing by me through good and bad times.

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# CHAPTER 1

# INTRODUCTION

The work done in this thesis is concerned about some soluble groups satisfing certain group properties. Chapter 2 is independent of the rest . The next chapters are closely related and have the same origin; a question of Paul Erdös.

Chapter 2 is initiated by several authors' papers on rewriting products of group elements and permutable subgroups in [$BL_1$], [$BL_2$], [$LMR_1$] and [$RW_1$]. It is based on the paper by P.S. Kim and A.H. Rhemtulla in [KR].

We will introduce notations and examples before the main results.

Let $n$ be a fixed positive integer, $X = \{x_1, \dots, x_n\}$ a set of $n$ symbols and $F = F(X)$ the free group on $X$. Let $U = \{u_1, u_2, \dots\}$ and $V = \{v_1, v_2, \dots\}$ be non-empty sets of elements in $F$. Define the class $P(U,V)$ to consist of groups $G$ such that given an $n$-tuple $(g_1, \dots, g_n)$ of elements in $G$, $u(g_1, \dots, g_n) = v(g_1, \dots, g_n)$ for some $u \in U$ and some $v \in V$, $v \neq u$. Some examples;

**(1.1)** Let $U = \{u\}$ where $u(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$ and $V = \{u_\sigma \mid \sigma \in S_n \backslash 1\}$ where $S_n$ is the symmetric group of degree $n$ and

$$u_\sigma(x_1, \dots, x_n) = u(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Then every group in $P(U,V)$ is finite-by-abelian-by-finite. Conversely every finite-by-abelian-by-finite group is in $P(U,V)$ for some suitable $n$. This was shown by Curzio, Longobardi, Maj and Robinson in [CLMR]. These groups are more commonly referred to as $P_n$-groups.

**(1.2)** Let $u = u(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$, $U = \{u_\sigma \mid \sigma \in S_n\}$ and $V = U$. Then $P(U,V)$-groups, more commonly referred to as $Q_n$-groups or rewritable groups, are again finite-by-abelian-by-finite groups as shown by Blyth in [$BL_1$]. That an abelian-by-finite group is in $Q_n$ for some $n$ is implicit in Theorem 1 of Kaplansky [KA].

**(1.3)** If we take $n = 2$, $u = u(x_1, x_2) = (x_1 x_2)^r$, $v = v(x_1, x_2) = (x_2 x_1)^r$ where $r > 0$ is fixed and $U = \{u\}$, $V = \{v\}$ then $G \in P(U,V)$ if and only if $G/Z(G)$ is of exponent $r$. If $G \in P(U,V)$, $x_1 = y^{-1}x$ and $x_2 = y$, then $y^{-1}x^r y = (y^{-1}xy)^r = x^r$ for all $y$ in $G$ and hence $x^r \in Z(G)$. Conversely suppose $G/Z(G)$ is of exponent $r$, then $(x_1 x_2)^r \in Z(G)$ and $(x_1 x_2)^r = x_1^{-1}(x_1 x_2)^r x_1 = (x_2 x_1)^r$.

In general, the classes $P(U,V)$ may be viewed as generalising varieties and, except for some specific sets $U$ and $V$, it is very difficult to describe them. We now turn to related classes of groups.

Let $n > 0$ be fixed, $X = \{x_1, \ldots, x_n\}$ be a set of idempotent variables and $S = S(X)$ the free semigroup generated by $X$. Thus for any $u \in S$, $u = u(x_1, \ldots, x_n) = x_{11} x_{12} \cdots x_{1m}$ where $x_{1i} \in X$ and $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. If $(H_1, \ldots, H_n)$ is an $n$-tuple of subgroups of a group $G$ then denote by $u(H_1, \ldots, H_n)$ the set $\{u(h_1, \ldots, h_n) \mid h_i \in H_i\}$. Thus if $u(x_1, \ldots, x_n)$ is as above, then

$$u(H_1, \ldots, H_n) = H_{11} H_{12} \ldots H_{1m}.$$

If $U$ and $V$ are sets of elements in $S$ then define the class $SP(U,V)$ to consist of groups $G$ such that for any $n$-tuple $(H_1, \ldots, H_n)$ of subgroup of $G$,

$u(H_1,\dots,H_n) = v(H_1,\dots,H_n)$ for some $u \in U$ and some $v \in V$, $v \neq u$. Some examples:

**(1.4)** Let $U = \{u\}$ where $u(x_1,\dots,x_n) = x_1x_2\cdots x_n$ and $V = \{u_\sigma \mid \sigma \in S_n \backslash 1\}$. As in (1.1), $u_\sigma(x_1,\dots,x_n) = u(x_{\sigma(1)},\dots,x_{\sigma(n)})$. Finitely generated soluble $SP(U,V)$ groups are finite-by-abelian. Conversely every finite-by-abelian group is an $SP(U,V)$ group for some integer $n$. These results are contained in [$RW_1$]. From [$LMR_1$] we know that periodic $SP(U,V)$-groups are locally finite.

**(1.5)** For each positive integer $r$, let $u_r = u_r(x,y) = (xy)^r$, and $v_r = v_r(x,y) = (yx)^r$. Let $U = \{u_r,\ r = 1,2,\dots\}$ and $V = \{v_r,\ r = 1,2,\dots\}$. Then the class $SP(U,V)$ is precisely the class of groups in which every subgroup is elliptically embedded. Groups with this property are considered in [$RW_2$] and [SM]. It is known that a finitely generated soluble group $G$ is in this class if and only if it is finite-by-nilpotent. The same is true if we replace "soluble groups" by "residually finite $p$-group" in the above statement. We show the two main results as follows in chapter 2;

**(1.6)Theorem.** *Let $U = \{u\}$ where $u$ is a word in idempotent variables $x_1,\dots,x_n$ for $n > 1$ and let $V = \{u_\sigma \mid \sigma \in S_n\}$. If $G$ is a finitely generated soluble group in $SP(U,V)$, then $G$ is nilpotent-by-finite.*

**(1.7)Theorem.** *Let $G$ be a finitely generated soluble group, $U = \{u\}$ where $u = u(x_1,\dots,x_n) = (x_1\cdots x_n)^r$ and $V = \{u_\sigma \mid \sigma \in S_n\}$ where $u_\sigma(x_1,\dots,x_n) = u(x_{\sigma(1)},\dots,x_{\sigma(n)})$. Then $G$ is an $SP(U,V)$-group for some $n > 1$, $r > 0$ if and only if $G$ is finite-by-nilpotent.*

Chapter 3 is motivated by a question of P. Erdös: If $G$ is a group such that subsets consisting of mutually noncommuting elements are all finite, then are they boundedly finite? B.H. Neumann [NE$_1$] answered this question affirmatively by proving that $G$ has the property stated above if and only if $G$ is centre-by-finite. Extensions of problems of this type are to be found in [GR] and [LW].

In addition, J.C. Lennox [LE] has studied bigenetic properties of finitely generated groups, that is, when all two-generator subgroups of a group $G$ has a group property $\wp$, $G$ also has $\wp$ for some classes of hyper-(abelian-by-finite) groups.

Contrarily, it is kmown that M.F. Newman has constructed an example (Unpublished. See [LE]) of a three-generator-infinite $p$-group ($p$ a prime) with all of its two-generator subgroups nilpotent. Therefore it is necessary for $G$ to be hyper-(abelian-by-finite) in the study of bigenetic properties.

In some classes of infinite groups, $G$ has $\wp$ if whenever we choose infinite sets $X_1, \ldots, X_n$ $(n \geq 1)$ of $G$, there exist $x_i \in X_i$ such that $\langle x_1, \ldots, x_n \rangle$ satisfies $\wp$. Some examples;

**(1.8)** Let $G$ be an infinite group. If for every pair $(X, Y)$ of infinite subsets of $G$, there exists $x$ in $X$ and $y$ in $Y$ such that $xy = yx$, then $G$ is abelian. From a result of B.H. Neumann [BN$_1$], it follows that $G$ is centre-by-finite so that the centre $Z$ of $G$ is infinite. For any $x, y$ in $G$ we take the infinite sets $Zx, Zy$ and by hypothesis, $[xz_1, yz_2] = [x, y] = 1$ for some $z_1, z_2$ in $Z$.

**(1.9)** Let $n > 1$ be a fixed integer and $G$ an infinite group. If every infinite subset $X$ of $G$ contains an element of order dividing $n$, then $G$ is of exponent dividing $n$. Let $F$ denote the $FC$-centre of $G$. This is the set of all elements

of $G$ having finitely many conjugates. If $x$ is in $G$ and $x^n \neq 1$, then the number of conjugates of $x$ must be finite and hence $x \in F$. Thus $C = C_G(x)$ is infinite since $|G : C|$ is finite. Let $D \subseteq C$ be the set of those elements in $C$ of order dividing $n$. Clearly $D$ is an infinite set and $(dx)^n = x^n \neq 1$ for all $d \in D$, contradicting the hypothesis.

With above examples in mind, it is proper to ask the extent to which a property for an infinite group $G$ is determined in the above fashion: Let $\mathcal{V}$ be a variety defined by the law $w(x_1, \ldots, x_n) = 1$ and assume that $n$ is the least number of variables required to determine $\mathcal{V}$. If $G$ is an infinite group such that whenever $X_1, \ldots, X_n$ are infinite subsets of $G$ there exists $x_i$ in $X_i$, $i = 1, \ldots, n$ such that $\langle x_1, \ldots, x_n \rangle$ is a $\mathcal{V}$-group. Does it follow that $G$ is a $\mathcal{V}$-group? It is too much to expect this to be true for all varieties $\mathcal{V}$ but we have no counter example.

In chapter 3, based on the paper by P.S. Kim, A.H. Rhemtulla and H. Smith [KRS], we proved the result when $\mathcal{V}$ is the class of metabelian groups.
P. Longobardi, M. Maj and A.H. Rhemtulla [LMR$_2$] proved the result when $\mathcal{V}$ is the class of nilpotent groups of class $n-1$. Their result is slightly stronger for they do not assume that $\langle x_1, \ldots, x_n \rangle$ is nilpotent of class $n-1$ but only that $[x_1, \ldots, x_n] = 1$.

From the point stated above, it is natural to ask a similiar type of a question: if $G$ is an infinite group such that whenever $X_1, X_2, X_3, X_4$ are any infinite subsets of $G$ there exists $x_i \in X_i$, $i = 1, 2, 3, 4$ such that $[[x_1, x_2], [x_3, x_4]] = 1$, then is $G$ metabelian?

At present we don't know the answer but we have affirmative answer for locally soluble groups in chapter 4.

Recently several authors in [LMRS] and [CLRW] have studied the group property when every infinite set of subgroups contains a pair that permute as subgroups. One of main results is that a finitely generated group $G$ is centre-by-finite if and only if every infinite set of subgroups of $G$ contains a pair that permute .

Using this result, we prove the following in chapter 5; A finitely generated group $G$ is quasi-Hamiltonian if every two infinite sets $\mathbf{X}$, $\mathbf{Y}$ of subgroups of $G$ contains $H \in \mathbf{X}$, $K \in \mathbf{Y}$ such that $HK = KH$.

## CHAPTER 2

## PERMUTABLE WORD PRODUCTS IN GROUPS

In this chapter, we shall not look at $P(U,V)$-groups, but concentrate our attention on $SP(U,V)$-groups. At present little is known about the various classes $SP(U,V)$ and $P(U,V)$. The reduction from soluble to nilpotent-by-finite in (1.6)Theorem will be achieved using several Lemmas. Most of Lemmas contain complicated computations. All results are from the paper by [KR].

We recall that a soluble group $G$ is said to be *minimax* if and only if it has a series $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ in which the factors are cyclic or quasicyclic. We shall call a group *constrained* if and only if there is no prime $p$ for which it has a section isomorphic to $C_p \wr C_\infty$, the standard restricted wreath product of a cyclic group of order $p$ by an infinite cyclic group. This terminology is due to P.H. Kropholler [KO], who gave a celebrated Theorem; *Every finitely generated constrained soluble group is minimax and hence it has finite rank.*

Since $SP(U,V)$ is subgroup and quotient closed, (2.1)Lemma implies that finitely generated soluble groups in the class of $SP(U,V)$ have finite rank.

**(2.1)Lemma.** *The wreath product of a cyclic group of order $p$ with the infinite cyclic group is not in the class $SP(U,V)$ where $U$, $V$ are as in the statement of (1.6)Theorem.*

**Proof.** Let $G$ be the wreath product of a cyclic group of order $p$ and an infinite cyclic group $\langle t \rangle$. Then we can identify each element of $G$ by a pair $(f(t),\ t^\alpha)$ where $f(t) \in \mathsf{F}_p\langle t \rangle$, the additive group of the group ring of the infinite cyclic group $\langle t \rangle$ over the field $\mathsf{F}_p$ of $p$ elements, and $\alpha \in \mathbb{Z}$. The

product of two such elements is then given by the rule: $(f(t),\ t^{\alpha})(g(t),\ t^{\beta}) = (f(t)+t^{\alpha}\cdot g(t),\ t^{\alpha+\beta})$. The elements of the base group correpond to those pairs where $\alpha = 0$ and the elements of the top group correspond to those pairs where $f(t) = 0$.

We are given $X = \{x_1,\dots,x_n\}$, $u = u(X) = x_{11}x_{12}\dots x_{1m}$; $x_{1i} \in X$, $x_{1i} \neq x_{1i+1}$, $i = 1,\dots,m-1$ and we are required to show that there exist subgroups $H_1,\dots,H_n$ of $G$ such that $H_{11}H_{12}\dots H_{1m} \neq H_{\phi(11)}H_{\phi(12)}\dots H_{\phi(1m)}$ for any $\phi \neq 1$ in the symmetric group of degree $n$.

Take $H_i = \langle h_i\rangle$ where $h_i = ((1-t^{\alpha_i})f_i,\ t^{\alpha_i})$; $f_i = f_i(t)$ and $\alpha_i$ are to be chosen appropriately. Note that $h_i^k = ((1-t^{k\alpha_i})f_i,\ t^{k\alpha_i})$ and a general element of $u(H_1,\dots,H_n)$ is $h_{11}^{k_1}\dots h_{1m}^{k_m} =$

$$((1-t^{k_1\alpha_{11}})f_{11},\ t^{k_1\alpha_{11}})\cdots((1-t^{k_m\alpha_{1m}})f_{1m},\ t^{k_m\alpha_{1m}}) =$$

$$((1-t^{k_1\alpha_{11}})f_{11}+t^{\lambda_1}(1-t^{k_2\alpha_{12}})f_{12}+\cdots+t^{\lambda_{m-1}}(1-t^{k_m\alpha_{1m}})f_{1m},\ t^{\lambda_m})$$

where $\lambda_i = k_1\alpha_{11}+\cdots+k_i\alpha_{1i}$, $i = 1,\dots,m$. Partition the set $\{1,\dots,m\}$ as the union $S_1\cup\cdots\cup S_n$ where $S_i = \{j \mid x_{1j} = x_i\}$. Then a general element of $u(H_1,\dots,H_n)$ is of the form

$$(\sum_{i=1}^{n}(f_i\sum_{j\in S_i}(t^{\lambda_{j-1}}-t^{\lambda_j})),\ t^{\lambda_m})$$

with the understanding that $\lambda_0 = 0$. Likewise $u(H_{\phi(1)},\dots,H_{\phi(n)})$ consists of elements of the form

$$(\sum_{i=1}^{n}(f_{\phi(i)}\sum_{j\in S_i}(t^{\mu_{j-1}}-t^{\mu_j})),\ t^{\mu_m})$$

where $\mu_i = \ell_1\alpha_{\phi(11)} + \cdots + \ell_i\alpha_{\phi(1i)}$, $i = 1, \ldots, m$ and $\mu_0 = 0$. If $\sigma$ denotes the inverse of $\phi$, then we may write these elements as

$$\Big(\sum_{i=1}^{n}\Big(f_i \sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})\Big),\ t^{\mu_m}\Big).$$

Now

$$\Big(\sum_{i=1}^{n}\Big(f_i \sum_{j\in S_i} (t^{\lambda_{j-1}} - t^{\lambda_j})\Big),\ t^{\lambda_m}\Big) = \Big(\sum_{i=1}^{n}\Big(f_i \sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})\Big),\ t^{\mu_m}\Big)$$

implies $\lambda_m = \mu_m$ and

$$\sum_{i=1}^{n} f_i\Big(\sum_{j\in S_i} (t^{\lambda_{j-1}} - t^{\lambda_j})\Big) = \sum_{i=1}^{n} f_i\Big(\sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})\Big). \tag{2.2}$$

Let $p_1, \ldots, p_n$ be distinct primes, each greater than $m$. Put $p = p_1 \ldots p_n$, $k_i = 1$ and $\alpha_i = p/p_i$, $i = 1, \ldots, n$. Then for each $i > 0$, $\lambda_i = \alpha_{11} + \cdots + \alpha_{1i}$. Let $f_i = t^{p^i}$. Note that $\lambda_j < p$ for all $j$ and they are all distinct. Also note that $f_i t^{\lambda_j} = f_{i'} t^{\mu_{j'}}$ implies $p^i + \lambda_j = p^{i'} + \mu_{j'}$. Hence $\mu_{j'} \equiv \lambda_j \mod p$ so that $\mu_{j'} \not\equiv \lambda_i \mod p$ for any $i \neq j$. Thus each $\lambda_j$ is congruent modulo $p$ to precisely one $\mu_{j'}$.

Now $1 \in S_{\sigma(k)}$ for some $k$. Thus $f_k(t^{\mu_0} - t^{\mu_1})$ is a term on the right hand side of (2.2). Since $\mu_0 = 0$ and the only $\lambda_i$ equal to zero is $\lambda_0$, $f_k(t^{\lambda_0} - t^{\lambda_1})$ appears on the left hand side of (2.2). In particular $1 \in S_k$. Since $S_1, \ldots, S_n$ partition the set $\{1, \ldots, m\}$ and $1 \in S_k \cap S_{\sigma(k)}$, it follows that $\sigma(k) = k$. Hence $\mu_1$ and $\lambda_1$ are both congruent to zero mod $p/p_k$; it follows that $\mu_1 = \lambda_1$.

Suppose, by way of induction, that we have established that $\mu_j = \lambda_j$ for all $j < e$. Then $\mu_e - \mu_{e-1} = \ell_e\alpha_{\phi(1e)}$ which is congruent to zero mod all primes $p_i$ except possibly one namely, $p_{\phi(1e)}$. Now we look at $\{\lambda_j - \lambda_{e-1},\ j = e, \ldots, m\}$. $\lambda_e - \lambda_{e-1} = \alpha_{1e}$ is congruent to zero mod all primes $p_i$, $p_i \neq p_{1e}$.

For each of the other $\lambda_j - \lambda_{e-1}$, we can find at least two primes amongst $\{p_1, \ldots, p_n\}$ such that $\lambda_j - \lambda_{e-1}$ is not congruent to zero mod either of them. Hence $0 \not\equiv \alpha_{1e} \equiv \ell_e \alpha_{\phi(1e)} \mod p_{1e}$. But $\alpha_{\phi(1e)} \neq \alpha_{1e}$ implies $\alpha_{\phi(1e)} \equiv 0 \mod p_{1e}$. Thus $\alpha_{\phi(1e)} = \alpha_{1e}$, and $x_{1e} = x_{\phi(1e)} = x_{k'}$, say. Thus $\sigma(k') = k'$ and $e \in S_{k'}$. Thus $f_{k'}(t^{\mu_{e-1}} - t^{\mu_e}) = f_{k'}(t^{\lambda_{e-1}} - t^{\lambda_e})$ and $\lambda_e = \mu_e$.

It is now clear that $\phi(j) = j$ for all $j = 1, \ldots, n$ and hence $\phi$ is the identity permutation of the set $\{1, \ldots, n\}$ as required. ♠

Let $C$ be a torsion-free abelian group and let $A \leq Aut(C)$, we extend the action of $A$ to the rational vector space $V = C \otimes_{\mathbf{Z}} \mathbf{Q}$ as natural way, where $\mathbf{Q}$ is the field of rational number.

$A$ is said to be *rationally irreducible* on $C$ if $C/B$ is periodic whenever $B$ is nontrivial $A$-admissible subgroup of $C$. Then it is easy to see that $A$ is rationally irreducible if and only if $A$ is irreducible as a group of linear transformation of $V$.

**(2.3)Lemma.** *Let $G = \langle A, t \rangle$ where $A$ is a torsion-free abelian group of finite rank on which $\langle t \rangle$ acts rationally irreducibly. If $G \in SP(U, V)$ where $U$, $V$ are as in the statement of (1.6) Theorem, then for some positive integer $k$, $\langle t^k \rangle$ acts trivially on $A$.*

**Proof.** We are given a group $G = \langle A, t \rangle$ where $A$ is torsion-free abelian of finite rank on which $\langle t \rangle$ acts rationally irreducibly. Let us assume, if possible, that $[A, t] \neq 1$. Then $V = A \otimes_{\mathbf{Z}} \mathbf{Q}$ is an irreducible $\mathbf{Q}\langle t \rangle$- module and by Schur's Lemma, the centralizer ring $\Gamma = End_{\mathbf{Q}\langle t \rangle} V$ is a division ring of finite dimension over $\mathbf{Q}$. The image of $\langle t \rangle$ in $End_{\mathbf{Q}} V$ clearly lies in and spans $\Gamma$ so that $\Gamma$ is an algebraic number field. Moreover, regarded as a $\Gamma$-space, $V$ is one

dimensional. Thus we may consider $A$ to be an additive subgroup of $\mathbf{Q}(\tau)$ for some algebraic number $\tau$ and the action of conjugation by $t$ as multiplication by $\tau$.

Let $h_i = b_i(1-\tau^{\alpha_i})t^{\alpha_i}$ for suitable integer $\alpha_i$ and $b_i(1-\tau^{\alpha_i}) \in A$. Let $H_i = \langle h_i \rangle$. Note that $h_i^k = b_i(1-\tau^{k\alpha_i})t^{k\alpha_i}$.

As in (2.1)Lemma, we are given $u = u(x_1, \dots, x_n) = x_{11}x_{12}\dots x_{1m}$; $x_{1i} \in \{x_1, \dots, x_n\}$, $x_{1i} \neq x_{1i+1}$, $i = 1, \dots, m-1$; and we need to show that with proper choice of $b_i$ and $\alpha_i$, the subgroups $H_1, \dots, H_n$ can be found such that $H_{11}\dots H_{1m} \neq H_{\phi(11)}\dots H_{\phi(1m)}$ for any $\phi \neq 1$ in $S_n$. Now

$$h_{11}^{k_1}\dots h_{1m}^{k_m} = b_{11}(1-\tau^{k_1\alpha_{11}})t^{k_1\alpha_{11}}\dots b_{1m}(1-\tau^{k_m\alpha_{1m}})t^{k_m\alpha_{1m}}$$

$$= b_{11}(1-\tau^{\lambda_1}) + b_{12}(\tau^{\lambda_1}-\tau^{\lambda_2}) + \dots + b_{1m}(\tau^{\lambda_{m-1}}-\tau^{\lambda_m})t^{\lambda_m}$$

where $\lambda_i = k_1\alpha_{11} + \dots + k_i\alpha_{1i}$, $i = 1, \dots, m$.

We shall put $\lambda_0 = 0$ and write $1 = \tau^0 = \tau^{\lambda_0}$.

Thus a general element of $u(H_1, \dots, H_n)$ has the form

$$\sum_{i=1}^{n}\Big(b_i \sum_{j\in S_i}(\tau^{\lambda_{j-1}} - \tau^{\lambda_j})\Big)t^{\lambda_m}$$

where $S_i = \{j;\ x_{1j} = x_i\}$ so that $\{1, \dots, m\}$ is the disjoint union of $S_1, \dots, S_n$. Likewise the general element of $u(H_{\phi(1)}\dots H_{\phi(n)})$ has the form

$$\sum_{i=1}^{n}\Big(b_i \sum_{j\in S_{\sigma(i)}}(\tau^{\mu_{j-1}} - \tau^{\mu_j})\Big)t^{\mu_m}$$

where $\mu_i = \ell_1\alpha_{\phi(1)} + \dots + \ell_i\alpha_{\phi(i)}$, $i = 1, \dots, m$; $\mu_0 = 0$ and $\sigma = \phi^{-1}$. This is shown in the same way as in the proof of (2.1)Lemma.

In particular $\mu_m = \lambda_m$ and

$$(2.4) \qquad \sum_{i=1}^{n} b_i \Big( \sum_{j \in S_i} (\tau^{\lambda_j - 1} - \tau^{\lambda_j}) - \sum_{j \in S_{\sigma(i)}} (\tau^{\mu_j - 1} - \tau^{\mu_j}) \Big) = 0.$$

Now we return to pick $b_i$ and $\alpha_i$ appropriately. For each integer $r > 1$, pick primes $p_{r1}, \ldots, p_{rn}$ to satisfy $2^r < p_{r1}$ and $p_{ri}^2 < p_{ri+1}$, $i = 1, \ldots, n-1$. Put $q_r = p_{r1} \cdots p_{rn}$, $b_{ri}(y) = y^{q_r^i}$ and $\alpha_{ri} = q_r / p_{ri}$, $i = 1, \ldots, n$. To make the notation simpler, we shall write $b_i$ for $b_{ri}$ and $\alpha_i$ for $\alpha_{ri}$, where there is no ambiguity. Since there are infinitely many choices of $q_r$ and each choice of $q_r$ determines the sequence $H_1, \ldots, H_n$ of subgroups which in turn corresponds to some permutation $\phi \neq 1$ such that $u(H_1, \ldots, H_n) = u(H_{\phi(1)}, \ldots, H_{\phi(n)})$, there is an infinite number of choices of $r$ such that $q_r$ correspond to the same permutation $\phi$.

If, for some value of $r$, we have the following stronger version of (2.4):

$$\sum_{i=1}^{n} b_i(y) L_i(y) - \sum_{i=1}^{n} b_i(y) M_i(y) = 0$$

where

$$L_i(y) = \sum_{j \in S_i} (y^{\lambda_j - 1} - y^{\lambda_j}), \quad M_i(y) = \sum_{j \in S_{\sigma(i)}} (y^{\mu_j - 1} - y^{\mu_j})$$

and $y$ is an indeterminant; then $\mu_j = \lambda_j$ for all $j$ and $\phi = 1$. This is seen using arguments similiar to those in the proof of (2.1)Lemma. Thus we may suppose that for every $r$,

$$P(y) = \sum_{i=1}^{n} b_i(y) L_i(y) - \sum_{i=1}^{n} b_i(y) M_i(y)$$

is not zero but $P(\tau) = 0$. If $Q(y)$ is any non-trivial segment of $P(y)$ such that $Q(\tau) = 0$, then $Q'(y) = P(y) - Q(y)$ is a segment of $P(y)$ with $Q'(\tau) = 0$.

Moreover one or both of $Q(y)$ or $Q'(y)$ contains at least as many monomials from $\sum_{i=1}^{n} b_i(y)L_i(y)$ as from $\sum_{i=1}^{n} b_i(y)M_i(y)$. Let $Q(y)$ be such a segment of $P(y)$ of shortest length. Thus

I. $Q(y) \neq 0$

II. $Q(\tau) = 0$

III. $Q(y)$ contains at least as many monomials from $\sum_{i=1}^{n} b_i(y)L_i(y)$ as from $\sum_{i=1}^{n} b_i(y)M_i(y)$ and

IV. No proper segment has properties I and II.

We write $Q(y) = Q_1(y) - Q_2(y)$ where $Q_1(y)$ is a segment $\sum_{i=1}^{u} \pm y^{\lambda_i'}$ of $\sum_{i=1}^{n} b_i(y)L_i(y)$, $Q_2(y)$ a segment $\sum_{i=1}^{v} \pm y^{\mu_i'}$ of $\sum_{i=1}^{n} b_i(y)M_i(y)$ and we may suppose that there is no term in $Q_1(y)$ equal to any term in $Q_2(y)$. If $Q_1(y)$ has only one term in it, then $Q(y) = \pm y^{\lambda}$ or $\pm y^{\lambda} \pm y^{\mu}$ where $0 \neq \lambda$ and $\mu \neq \lambda$. In both cases $Q(\tau) = 0$ implies $\tau$ is a root of unity and $[t^k, A] = 1$ for some $k > 0$, as required.

We may therefore assume that $Q_1(y) = \sum_{i=1}^{u} \pm y^{\lambda_i'}$ has more than one term; $\lambda_i' = q^{i'} + \lambda_{ji}$ and $0 \leq \lambda_1' < \cdots < \lambda_u'$. Similiarly $Q_2(y) = \sum_{i=1}^{v} \pm y^{\mu_i'}$ where $\mu_1' \leq \cdots \leq \mu_v'$. Let

$$\nu_1 = \min\{\lambda_1', \mu_1'\} \quad \text{and} \quad \nu_2 = \max\{\lambda_u', \mu_v'\}$$

Then $y^{-\nu_1} \cdot Q(y)$ is a polynomial of degree $\nu_2 - \nu_1$ with non-zero constant term. Moreover $\nu_2 - \nu_1 \geq \lambda_u' - \lambda_1' > 2^r$. Thus the degree of the polynomial increases with $r$ and hence there are infinitely many expressions

$$1 = \sum_{i=1}^{u+v} \varepsilon_i \tau^{\gamma_i}$$

where $\gamma_i \geq 0$, $\varepsilon_i \in \{-1,0,1\}$ and no subsum of the right hand side of the equation is zero. But this is not possible by Theorem 1 of [PO] which we state below for convenience. This completes the proof. ♠

**(Van Der Poorten)Theorem.** *Let $K$ be a field of characteristic zero and $H$ a finitely generated subgroup of the multiplicative group of $K$. Then for each integer $m > 0$ there are only finitely many relations $u_1 + \cdots + u_m = 1$ with each $u_i \in H$ and no subsum of the left hand side is zero.*

**(2.5)Lemma.** *If $G = \langle A, t\rangle$, where $A \trianglelefteq G$ and is abelian of finite rank, and $G \in SP(U,V)$ where $U$, $V$ are as in the statement of (1.6)Theorem, then for some $\ell > 0$, $\langle A, t^\ell\rangle$ has a non-trivial centre.*

**Proof.** If the torsion subgroup of $A$ is non-trivial then it has a non-trivial subgroup $A_1$ of exponent $p$ for some prime $p$. This is finite since $A$ has finite rank, hence it is centralized by $t^\ell$ for some $\ell > 0$ and $A_1$ lies in the centre of $\langle A, t^\ell\rangle$. We may thus assume that $A$ is torsion-free. Let $D$ be a non-trivial subgroup of $A$ of least rank subject to $D \trianglelefteq G$. (2.3)Lemma applies to $\langle D, t\rangle$ and we conclude that $\langle D, t^k\rangle$ is abelian for some $k > 0$. Hence $D$ lies in the centre of $\langle A, t^k\rangle$. ♠

**(2.6)Lemma.** *Let $G = \langle A, t\rangle$, where $A$ is a torsion-free abelian group of finite rank on which $\langle t\rangle$ acts rationally irreducibly. If $G \in SP(U,V)$ where $U$, $V$ are as in the statement of (1.7)Theorem, then $\langle t\rangle$ acts trivially on $A$.*

**proof.** As the hypothesis of (1.7)Theorem is stronger than that of (1.6)Theorem , (2.3)Lemma and its proof applies. We follow the proof of (2.3)Lemma and reach the situation where we may assume $A$ to be an additive subgroup of $Q(\tau)$

for some algebraic number $\tau$ and the action of $t$ under conjugation is that of multiplication by $\tau$. Furthermore we may assume $\tau$ to a primitive $k$th root of unity and we need to show that $\tau = 1$.

Let $h_i = (k^i(1-\tau),\ t^{-1})$ and $H_i = \langle h_i \rangle$. Observe that $h_i^\lambda = (k^i(1-\tau^\lambda),\ t^{-\lambda})$ and $h_i^k = t^{-k}$. Let $X = (H_1 \cdots H_n)^r$ and suppose that for some $\phi \neq 1$ in $S_n$, $X = (H_{\phi(1)} \cdots H_{\phi(n)})^r$. If $\phi(1) \neq 1$, then $\langle H_1, H_{\phi(1)} \rangle \subseteq X$. But this is not possible since $X$ is the union of a finite number of cosets of $\langle t^k \rangle$ whereas $\langle H_1, H_{\phi(1)} \rangle$ contains the subgroup generated by $(k^{\phi(1)} - k)(1-\tau)$, an infinite cyclic subgroup of $Q(\tau)$, not contained in any finite union of cosets of $\langle t^k \rangle$. Hence $\phi(1) = 1$ and similiarly $\phi(n) = n$.

For any permutation $\pi$ in $S_n$, a typical element $x$ of $H_{\pi(1)} \cdots H_{\pi(n)}$ has the form

$$\begin{aligned} x &= h_{\pi(1)}^{\lambda_1} \cdots h_{\pi(n)}^{\lambda_n} \\ &= \left(k^{\pi(1)}(1-\tau^{\lambda_1}) + k^{\pi(2)}(1-\tau^{\lambda_2})\tau^{\lambda_1} + \cdots + k^{\pi(n)}(1-\tau^{\lambda_n})\tau^{\lambda_1+\cdots+\lambda_{n-1}},\ t^{-\mu}\right) \end{aligned}$$

where $\mu = \lambda_1 + \cdots + \lambda_n$ and $\lambda_i$ are arbitrary integers. In turn these elements may be written as

$$\left(k^{\pi(1)}(1-\tau^{\alpha_1}) + k^{\pi(2)}(\tau^{\alpha_1} - \tau^{\alpha_2}) + \cdots + k^{\pi(n)}(\tau^{\alpha_{n-1}} - \tau^{\alpha_n}),\ t^{-\alpha_n}\right)$$

where $\alpha_i$ are arbitrary integers.

In particular $t^{-\alpha_0}x$ has the form $(h,\ t^{-\alpha_n})$ where

$$\begin{aligned} h &= k^{\pi(1)}(\tau^{\alpha_0} - \tau^{\alpha_1}) + k^{\pi(2)}(\tau^{\alpha_1} - \tau^{\alpha_2}) + \cdots + k^{\pi(n)}(\tau^{\alpha_{n-1}} - \tau^{\alpha_n}) \\ &= k^{\pi(1)}\tau^{\alpha_0} + \tau^{\alpha_1}(k^{\pi(2)} - k^{\pi(1)}) + \cdots + \tau^{\alpha_{n-1}}(k^{\pi(n)} - k^{\pi(n-1)}) - \tau^{\alpha_n}k^{\pi(n)}. \end{aligned}$$

Thus, if $\alpha_0$ is a given fixed integer, the real part of $h$ is maximised by choosing $\alpha_i \equiv 0 \mod k$ if $\pi(i+1) > \pi(i)$, $\alpha_i \equiv q \mod k$ where $q = [k/2]$ if $\pi(i+1) < \pi(i)$; $i = 1, \ldots, n-1$ and $\alpha_n \equiv q \mod k$. If $\pi$ is the identity permutation then this value is

$$k\cos(2\pi\alpha_0/k) + (k^n - k) - k^n\cos(2\pi q/k).$$

On the other hand if $\pi \neq 1$ and $\pi(1) = 1, \pi(n) = n$, then the maximum real part of the value of $h$ is

$$k\cos\frac{2\pi\alpha_0}{k} + (k^n - k) - k^n\cos\frac{2\pi q}{k} + \sum(k^{\pi(i)} - k^{\pi(i+1)})(1 - \cos\frac{2\pi q}{k})$$

where the sum is over all values of $i$ such that $\pi(i) > \pi(i+1)$. This value is clearly greater than the value obtained for the identity permutation $\pi$.

Now the general element of $(H_{\pi(1)} \ldots H_{\pi(n)})^r$ is $(h, t^{-\alpha})$ where $h$ is expressible in the form

$$\sum_{i=1}^{r} k^{\pi(1)}\tau^{\alpha_{i0}} + \tau^{\alpha_{i1}}(k^{\pi(2)} - k^{\pi(1)}) + \cdots + \tau^{\alpha_{in-1}}(k^{\pi(n)} - k^{\pi(n-1)}) - \tau^{\alpha_{in}}k^{\pi(n)}$$

where $\alpha_{10} = 0$, $\alpha_{in} = \alpha_{i+10}$, $i = 1, \ldots, r-1$, $\alpha_{rn} = \alpha$. By picking the values for $\alpha_{ij}$ to maximise the real part of $h$ as above, it is clear that the value achieved when $\pi \neq 1$ is greater than for $\pi = 1$. Thus $(H_{\phi(1)} \ldots H_{\phi(n)})^r$, $\phi(1) = 1$, $\phi(n) = n$, $\phi \neq 1$, contains elements not contained in $(H_1 \ldots H_n)^r$. This completes the proof. ♠

For convenience, we write main Theorems from Introduction.

**(1.6)Theorem.** *Let* $U = \{u\}$ *where* $u$ *is a word in idempotent variables* $x_1, \ldots, x_n, n > 1$; *and let* $V = \{u_\sigma \mid \sigma \in S_n\}$. *If* $G$ *is a finitely generated soluble group in* $SP(U,V)$, *then* $G$ *is nilpotent-by-finite.*

**Proof.** By hypothesis, $X = \{x_1, \ldots, x_n\}$ $u = u(X) = x_{11}x_{12}\ldots x_{1m}$ where $x_{1i} \in X$ for all $i = 1, \ldots, m$ and $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. Let $G$ be a finitely generated soluble group such that for any $n$-tuple $(H_1, \ldots, H_n)$ of subgroups of $G$, there is a permutation $\sigma \neq 1$ in $S_n$ such that

$$u(H_1, \ldots, H_n) = H_{11}H_{12}\ldots H_{1m} = u(H_{\sigma(1)}, \ldots, H_{\sigma(n)}) = H_{\sigma(11)}\ldots H_{\sigma(1m)}.$$

We need to show that $G$ is nilpotent-by-finite, and we proceed by induction on the solubility length of $G$. If $G$ is abelian then there is nothing to prove. Let $G$ be soluble of length $d$ and assume that the result holds for soluble groups of smaller length. Since the class $SP(U,V)$ is subgroup and quotient closed, we may suppose that $G$ has a normal abelian subgroup $A$ such that $G/A$ is nilpotent-by-finite. In particular $G$ is abelian-by-polycyclic. By (2.1)Lemma, $G$ has finite rank.

As $G$ is finitely generated abelian-by-polycyclic, it satisfies the maximal condition for normal subgroups. If $G$ is not nilpotent-by-finite, then let $B$ be a maximal normal subgroup of $G$ such that $G/B$ is not nilpotent-by-finite. Now we replace $G$ by $G/B$ and hence assume that every proper quotient of $G$ is nilpotent-by-finite.

Let $T$ be the torsion subgroup of $A$. Then $T$ has finite rank and is of bounded exponent since $G$ satisfies the maximal condition for normal subgroups. Thus $T$ is finite, and $C = C_G(T)$, the centraliser of $T$ in $G$, is of finite index

in $G$. If $T \neq 1$ then $G/T$ is nilpotent-by-finite and hence $C/T$ is nilpotent-by-finite. Since $T \leq Z(C)$, the centre of $C$, then $C$ and hence $G$ would be nilpotent-by-finite. Thus we assume $T = 1$ and hence $A$ is torsion-free, and by passing to a suitable subgroup of finite index in $G$, if necessary, we may assume further that $G/A$ is a finitely generated torsion-free nilpotent group. Thus there exists a finite set $T = \{t_1, \ldots, t_r\}$ of elements in $G$ such that $G = \langle A, T\rangle$ and

$$A = G_0 \leq \langle G_0, t_1\rangle = G_1 \leq \ldots \leq \langle G_{r-1}, t_r\rangle = G_r = G$$

is a central series from $A$ to $G$ with torsion-free factors.

If $r = 1$ then $G = \langle A, t_1\rangle$. By (2.5)Lemma $Z\big(\langle A, t_1^{\ell_1}\rangle\big) \neq 1$ for some $\ell_1 > 0$ and hence $D = A \cap Z\big(\langle A, t_1^{\ell_1}\rangle\big)$ is a non-trivial normal subgroup of $G$. By our choice of $G, G/D$ is nilpotent-by-finite and hence $G$ is nilpotent-by-finite.

Now suppose we have established the result for the case $r < d$ and suppose $r = d$. By the induction hypothesis, $G_{d-1}$ is nilpotent-by-finite and $G = \langle G_{d-1}, t_d\rangle$. Let $H = \langle A, G_{d-1}^{\ell}\rangle$ for some suitable $\ell > 0$ so that $H$ is nilpotent. Let $Y = A \cap Z(H)$ then $Y$ is normal in $\langle H, t_d\rangle$ which is of finite index in $G$. Moreover $Z\big(\langle Y, t_d^{\ell_1}\rangle\big) \neq 1$ for some $\ell_1 > 0$ by (2.5)Lemma, so that $D_1 = Y \cap Z\big(\langle Y, t_d^{\ell_1}\rangle\big)$ is a non-trivial subgroup of $G$ contained in the centre of $\langle H, t_d^{\ell_1}\rangle$ which is of finite index in $G$. We may replace $\langle H, t_d^{\ell_1}\rangle$ by its normal interior in $G$, if necessary; it still contains $A$ and hence $D_1$. Now $\langle H, t_d^{\ell_1}\rangle/D_1$ is nilpotent-by-finite, $D_1 \leq Z\big(\langle H, t_d^{\ell_1}\rangle\big)$ and $\langle H, t_d^{\ell_1}\rangle$ is of finite index in $G$. Thus $G$ is nilpotent-by-finite, as required. ♠

In the case of finitely generated linear $SP(U, V)$-group, (1.6)Theorem follow from Tits'Theorem (See[WE]); ***A finitely generated linear group either is soluble-by-finite or contains a non-cyclic free subgroup.***

The latter does not occur in $SP(U,V)$-groups and hence we have the following immediately.

**(2.7)Corollary.** *A finitely generated linear $SP(U,V)$-group is nilpotent-by-finite where $U$, $V$ are as in the statement of (1.6)Theorem.*

**(1.7)Theorem.** *Let $G$ be a finitely generated soluble group, $U = \{u\}$ where $u = u(x_1, \ldots, x_n) = (x_1 \cdots x_n)^r$ and $V = \{u_\sigma \mid \sigma \in S_n\}$ where $u_\sigma(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then $G$ is an $SP(U,V)$-group for some $n > 1$, $r > 0$ if and only if $G$ is finite-by-nilpotent.*

**Poof.** Since the hypothesis of (1.6)Theorem are satisfied by the group of (1.7)Theorem, we may assume $G$ to be finitely generated nilpotent-by-finite. Let $T$ be the maximal finite normal subgroup of $G$. Since we wish to show that $G$ is finite-by-nilpotent we may look at $G/T$, if necessary, and hence assume that $G$ has no non-trivial finite normal subgroup. Let $F$ be the Fitting subgroup of $G$. If $F \neq G$ then pick any $t \in G \backslash F$ such that $t^p \in F$. Clearly it is sufficient to show that $\langle F, t \rangle$ is nilpotent for $G/F$ is finite and soluble, we can reach $G$ from $F$ by a subnormal series with factors of prime order. Thus we assume $G = \langle F, t \rangle$, $t^p \in F$ and $F$ is torsion-free.

Let $H$ be the hypercentre of $G$. Then $H \cap F$ is isolated in $F$. This may be seen by first checking it for $Z(G) \cap F$ and then by taking the quotient of $G$ by this subgroup, and using induction. Observe that if $H \not\leq F$ then $G = HF$ and $G$ is nilpotent. So assume $H \leq F$. Next we look at $G/H$. If $G/H$ is nilpotent then so is $G$. So we assume $H = 1$. Let $A$ be a non-trivial normal subgroup of $G$ of least Hirsch length and $A \leq Z(F)$. Since $\langle A, t \rangle \in SP(U,V)$, $\langle t \rangle$ acts trivially on $A$ by (2.6)Lemma. Thus $A \leq Z(G)$ contradicting the assumption

that $Z(G) = 1$. This concludes the proof that if $G \in SP(U,V)$ then $G$ is finite-by-nilpotent.

Now suppose that $G$ is a finitely generated finite-by-nilpotent group. For any subgroup $L$ of $G$ let $\gamma(L)$ denote the nilpotent residual of $L$. Thus $\gamma(L)$ is the intersection of the terms of the lower central series of $L$. Let $F = \gamma(G)$. It is finite by hypothesis and $G/F$ is nilpotent of class $c_1$ for some $c_1 > 0$. Thus $\gamma(L) = \gamma_c(L)$ for all $L \leq G$ where $c = |F| + c_1$. We show, by induction on $|\gamma\langle H,K\rangle| = s$, that $(HK)^{d_s} = (KH)^{d_s} = \langle H,K\rangle$ for all subgroups $H, K$ of $G$ where $d_1 = (4r)^c$, $r =$ rank of $G$; $d_i = d_{i-1} + 2i(i + d_1)$, $i > 1$. In particular $(HK)^d = (KH)^d = \langle H,K\rangle$ for all $H, K$ where $d = d_f$ and $f = |F|$.

By Proposition 2 of [RW$_2$] $\Gamma(HK)^t = \Gamma(KH)^t = \langle K,H\rangle$ where $\Gamma = \gamma\langle K,H\rangle, t = (4r)^c$, and $r$ is the rank of $G$. Thus if $\Gamma = 1$ then $d_1 = t$ will suffice.

For any $a \in (HK)^t$, $a = gb$ for some $g \in \Gamma$ and $b \in (KH)^t$ so that $ab^{-1} = g \in \Gamma \cap (HK)^{2t}$. If $\Gamma \cap (KH)^{2t} = 1$, then $a = b$ and $(HK)^t = (KH)^t$. This implies $\langle H,K\rangle = (HK)^t$, and again $d_1 = t$ suffices.

If $\Gamma_1 = \Gamma \cap (HK)^{2t} \neq 1$, then for each integer $m \geq 1$ let $\Gamma_{m+1} = \Gamma_m \cap \Gamma_m^{HK}$ so that $\Gamma_m \subseteq (HK)^{2t+2m}$. Observe that $\Gamma_m = \Gamma_{m+1}$ implies $\langle\Gamma_m\rangle = \langle\Gamma_m^H\rangle = \langle\Gamma_m^K\rangle$. Since $\Gamma_m \subseteq \Gamma$ and $|\Gamma| = s$, $\Gamma_s = \Gamma_{s+1}$. Also note that $\langle\Gamma_m\rangle \subseteq \Gamma_m^s$. Thus the normal closure $N$ of $\Gamma_1$ in $\langle H,K\rangle$ lies in $(HK)^\lambda$ where $\lambda = \lambda_s = 2(s^2 + ts)$.

Now $NH$ and $NK$ both lie in $(HK)^\lambda$ and $(NHNK)^m \subseteq (HK)^{\lambda+m}$ for all $m > 0$. Rank of $\langle H,K\rangle/N$ is no greater than $r$, $\gamma(\langle H,K\rangle/N) =$

$\gamma_c(\langle H,K\rangle/N)$ and $|\gamma(\langle H,K\rangle/N)| < |\gamma(\langle H,K\rangle)|$. Thus by the induction hypothesis, $N(HK)^{d'} = N(KH)^{d'} = \langle H,K\rangle$ where $d' = t+\lambda_2+\cdots+\lambda_{s-1} = d_{s-1}$. Since $(HK)^\lambda \geq N$, we obtain $(HK)^{d_s} = (KH)^{d_s} = \langle H,K\rangle$ where $d_s = d_{s-1} + \lambda_s$.

Now that we have shown that for a finitely generated finite-by-nilpotent group $G$ there is an integer $d$ such that $(HK)^d = (KH)^d$ for all subgroups $H,K$ of $G$, we let $u = u(x,y) = (xy)^d$, $v = v(x,y) = (yx)^d$ then $G \in SP(U,V)$ where $U = \{u\}$, $V = \{u_\sigma \mid \sigma \in S_2\}$. This completes the proof of the second part of the theorem. ♠

We can not replace "nilpotent-by-finite" in (1.6)theorem by the stronger condition "finite-by-nilpotent" of (1.7)Theorem. It is tedious, but we will show that the infinite dihedral group $D_\infty$ lies in $SP(U,V)$ where $U = \{u\}$, $V = \{u_\sigma|\sigma \in S_n\}$ and $u = u(x_1,x_2,x_3,x_4) = x_1x_4x_2x_3x_2x_3x_4x_1$ and it is well-known that $D_\infty$ is not finite-by-nilpotent.

**(2.8)Example.** Let $U = \{u\}$ where $u = u(x_1,\ldots,x_4) = x_1x_4x_2x_3x_2x_3x_4x_1$ and $V = \{u_\sigma \mid \sigma \in S_4\}$. Then the infinite dihedral group $G$ is in $SP(U,V)$.

Consider $u(H_1,H_2,H_3,H_4)$ for given subgroups $H_1,H_2,H_3,H_4$ of $G$. If $H_1$ or $H_4$ is normal in $G$ then $u(H_1,H_2,H_3,H_4) = u(H_4,H_2,H_3,H_1)$. If $H_2$ or $H_3$ is normal in $G$ then $u(H_1,H_2,H_3,H_4) = u(H_1,H_3,H_2,H_4)$. So assume none of the $H_i$'s is normal in $G$. If for some $i$, $H_i$ is not of order two, then it contains a subgroup $K_i$ normal in $G$ and of index two in $H_i$. Moreover $u(H_{\sigma(1)}\ldots H_{\sigma(4)}) = K_iu(H_{\sigma(1)}\ldots H_{\sigma(4)})$ for all $\sigma \in S_n$, and we may replace $G$ by $G/K_i$ and each of $H_j$ by $H_jK_i/K_i$. Thus the essential case to be considered is one where each $H_i$ is of order two.

Now $G = \langle a, t \rangle$ where $a^t = a^{-1}$ and $t^2 = 1$. $H_i = \langle a^{\lambda_i} t \rangle$, $i = 1, 2, 3, 4$. We will show that the set $L = H_4 H_2 H_3 H_2 H_3 H_4$ equals the set $R = H_4 H_3 H_2 H_3 H_2 H_4$. From this it follows that $u(H_1, H_2, H_3, H_4) = H_1 L H_1 = H_1 R H_1 = u(H_1, H_3, H_2, H_4)$.

Now $a^\lambda \in H_2 H_3 H_2 H_3$ if and only if $\lambda = 0, \lambda_2 - \lambda_3, \lambda_3 - \lambda_2$ or $2\lambda_2 - 2\lambda_3$. $a^\lambda t \in H_2 H_3 H_2 H_3$ if and only if $\lambda = \lambda_2, \lambda_3, 2\lambda_2 - \lambda_3$ or $2\lambda_3 - \lambda_2$. Hence $H_2 H_3 H_2 H_3 \backslash H_3 H_2 H_3 H_2$ consists of $a^{2\lambda_2 - 2\lambda_3}$ only and $H_3 H_2 H_3 H_2 \backslash H_2 H_3 H_2 H_3$ consists of $a^{2\lambda_3 - 2\lambda_2}$ only.

But $H_4 a^{2\lambda_2 - 2\lambda_3} H_4$ consists of $a^\lambda$ where $\lambda \in \{2\lambda_2 - 2\lambda_3, 2\lambda_3 - 2\lambda_2\}$ and $a^\lambda t$ where $\lambda \in \{\lambda_4 - 2\lambda_2 + 2\lambda_3, \lambda_4 + 2\lambda_2 - 2\lambda_3\}$. From the symmetry between $\lambda_2$ and $\lambda_3$ above it is clear that $H_4 a^{2\lambda_2 - 2\lambda_3} H_4 = H_4 a^{2\lambda_3 - 2\lambda_2} H_4$. Thus the sets $L$ and $R$ are equal and $G \in SP(U, V)$.

We have not tried to analyse conditions on words $u$ for which $D_\infty \notin SP(U, V)$ where $U = \{u\}$ and $V = \{u_\sigma \mid \sigma \in S_n\}$.

## CHAPTER 3

## A CHARACTERIZATION OF INFINITE METABELIAN GROUPS

Throughout this chapter, $G$ is an infinite group . We write

$G \in \mathfrak{A}^2$ when $G$ is metabelian.

$G \in \mathfrak{A}^2_*$ whenever $X_1, X_2, X_3, X_4$ are infinite subsets of $G$ there exist $x_i \in X_i$ for $i = 1, 2, 3, 4$ such that $\langle x_1, x_2, x_3, x_4 \rangle$ is metabelian.

In the study of $\mathfrak{A}^2_*$-groups, **Ramsey's Theorem** occurs. Let $A$ be the family of 4-element sets of $G$ whose elements generate a metabelian subgroup and $B$ be the family of 4-element sets of $G$ whose elements generate a non-metabelian subgroup. Then $A \cup B$ is the family of all 4-element sets of $G$ and then there exists an infinite subset $S$ of $G$ such that either all four element subsets of $S$ belong to $A$ or all four element subsets of $S$ belong to $B$. By the property of $G \in \mathfrak{A}^2_*$, the latter can not happen. Suppose $S = \{s_1, s_2, \dots\}$ and $Z = \{[s_i, s_j] \mid i \neq j\}$. If $Z$ is infinite then $\langle Z \rangle$ is an infinite abelian subgroup of $G$. Suppose, on the other hand, that $Z = \{z_1, \dots, z_k\}$. For each $\ell = 1, \dots, k$, let $U_\ell$ denote the set of all $\{s_i, s_j\}$ such that $[s_i, s_j]$, for $i \leq j$, is equal to $z_\ell$. By Ramsey's Theorem once more, there is an integer $i \leq j$ and an infinite subset $S_1$ of $S$ such that, for all $s_i, s_j$ in $S_1$ with $i \leq j$, $[s_i, s_j] = z_\ell$. We may assume $S = S_1$. Then, for $i \leq j \leq k$, $[s_i, s_j s_k^{-1}] = [s_i, s_k^{-1}][s_i, s_j]^{s_k^{-1}} = [s_i, s_k^{-1}][s_i, s_k]^{s_k^{-1}} = 1$. In particular, for distinct $i, j, k, \ell$, with $i \leq j$ and $k \leq \ell$, $s_i s_j^{-1}$ and $s_k s_\ell^{-1}$ commute. If there is an infinite set of distinct elements $s_i s_j^{-1}$ which commute pairwise then we again have an infinite abelian subgroup. Suppose, on the contrary (by means of a relabelling if necessary), that $T = \{s_1 s_2^{-1}, s_3 s_4^{-1}, \dots, s_{N-1} s_N^{-1}\}$ is a maximal

set of pairwise commuting elements of this type. Choose $i \geq N$. Then the set $Y = \{s_i s_{i+1}^{-1}, s_i s_{i+2}^{-1}, \dots\}$ is certainly infinite and so some element $y$ of $Y$ does not lie in $T$. Clearly $y$ commutes with every element of $T$, and we have a contradiction which allows us to conclude that there is indeed an infinite abelian subgroup of $G$.

**(3.1)Lemma.** *$G \in \mathfrak{A}_*^2$ is metabelian if $G$ is one of the following.*

(1) *$G$ has the infinite centre*

(2) *$G$ is an $FC$-group*

(3) *$G$ is residually finite.*

**Proof.** Let $Z$ be the centre of $G$. For any $a, b, x, y$ in $G$, let $A = aZ = \{az \mid z \in Z\}$, $B = bZ$, $X = xZ$, $Y = yZ$. Then these four infinite subsets give the desired result in (1).

For (2), if $G$ is a periodic $FC$-group then $C_G(a) \cap C_G(b) \cap C_G(x) \cap C_G(y)$ has an infinite abelian subgroup $S$ and four subsets $aS$, $bS$, $xS$, $yS$ yield $[[a, b], [x, y]] = 1$. If $G$ is not periodic then we may assume $G$ is a finitely generated $FC$-group. In that case $G$ has the infinite centre.

For (3), let $N_i$ $(i \in I)$ be subgroups of finite index with $\cap_{i \in I} N_i = 1$, then $[[a, b], [x, y]] \equiv 1 \mod N_i$ for all $i$. So $[[a, b], [x, y]] = 1$ and $G$ is metabelian.

**(3.2)Theorem.** *Any group $G \in \mathfrak{A}_*^2$ having an element $x$ of infinite order is metabelian.*

**Proof.** Clearly we may assume that $G$ is finitely generated.

For any $a, b$ in $G$, consider sets $\{ax^i \mid i \in \mathbf{Z}\}$, $\{bx^i \mid i \in \mathbf{Z}\}$, $\{x^p \mid p : \text{primes}\}$, $\{x^q \mid q : \text{primes}, q \neq p\}$. These infinite sets yield $\langle a, b, x\rangle \in \mathfrak{A}^2$. Hence $[\langle x\rangle, G]$

is abelian. Let $J$ be $[\langle x\rangle, G]$ and $C$ be the centralizer of $\langle x\rangle J$ in $G$. If $J$ is finite, then $\langle x\rangle J$ is in $FC$-centre of $G$. Hence $|G : C|$ is finite and $C$ is finitely generated. Let $c_1, \ldots, c_n$ be generators of $C$. Then for each $c_i$ there exist $n_i$, $\ell_i$ such that $[x^{\ell_i+n_i}, c_i] = [x^{n_i}, c_i]$. Therefore $[x^{\ell_i}, c_i] = 1$ and $[x^{\ell}, C] = 1$ for some $\ell$, for example $\ell = \ell_1 \cdots \ell_n$. Hence the centre of $C$ is infinite and $C \in \mathfrak{A}^2$. Thus $C$ and $G$ are residually finite (See Theorem 1 [PH]) and $G \in \mathfrak{A}^2$.

We may take $J$ to be infinite. Pick any $a, b, c$ in $G$ and let $S = \{s \in J \mid \langle s, ua, vb, wc\rangle \in \mathfrak{A}^2$ for some $u, v, w \in J\}$. Then by $G \in \mathfrak{A}^2_*$, $J \backslash S$ is finite and $\langle S\rangle = J$. Since $J$ is a normal abelian subgroup, $[vb, wc] = [b, c]z$ for some $z \in J$ and $[s, ua] = [s, a]$. Hence $[[b, c], [s, a]] = 1$ for all $s \in S$ and $[[b, c], [J, a]] = 1$. Therefore $[G', [J, G]] = 1$.

If $[J, G]$ is finite, then let $D$ be the centralizer of $[J, G]$ in $G$ so that $[J, D, D] = 1$. Hence $[D', J] = 1$ by the Three Subgroup Lemma (See Lemma 2.13 [$RO_1$]). For any $a, b, c, d \in D$ and $u, v, w, z \in J$, $[[ua, vb], [wc, zd]] = 1$ implies $[[a, b], [c, d]] = 1$. Thus $D \in \mathfrak{A}^2$ and hence $D$ and $G$ are residually finite, so $G \in \mathfrak{A}^2$.

If $[J, G]$ is infinite, then for any $a, b, c, d$ in $G$ and $u, v, w, z$ in $[J, G]$, $[[ua, vb], [wc, zd]] = [[a, b], [c, d]]$ and hence $G \in \mathfrak{A}^2$. ♠

**Remark.** In the study of a torsion-free group $G \in \mathfrak{A}^2_*$, it is observed that for any $a$, $b$, $c$, in $G$, the sets $\{ac^i \mid i \in \mathbf{Z}\}$, $\{bc^i \mid i \in \mathbf{Z}\}$, $\{c^p \mid p$ : primes$\}$, $\{c^q \mid q$ : primes, $q \neq p\}$ yield $\langle a, b, c\rangle \in \mathfrak{A}^2$ and hence $G$ is 3-metabelian group. This special group has been studied by I.D. MacDonald [MA] and has the properties

(1) $G/Z(G)$ is metabelian (2) $G''$ has exponent 2,

hence $G'' = 1$ and $G$ is metabelian.

From now on, we confine our attention to periodic groups with $\mathfrak{A}_*^2$

**(3.3)Theorem.** *If $G \in \mathfrak{A}_*^2$ is periodic and soluble, then $G$ is metabelian.*

**Proof.** If $G'$ is finite, then $G$ is an $FC$-group and hence $G \in \mathfrak{A}^2$ by (3.1)Lemma.

We may assume $G'$ is infinite metabelian by induction on the derived length of $G$. If $G \notin \mathfrak{A}^2$, then there exists a finite subgroup $F$ which is not metabelian since $G$ is locally finite. By replacing $G$ by $G'F$, we may take $G = HF$ where $H$ is an infinite metabelian normal subgroup and $F$ is a finite subgroup with $F \notin \mathfrak{A}^2$. We note that if $N$ is any infinite normal subgroup of $G \in \mathfrak{A}_*^2$, then for any $a.b, c, d \in G$, the four sets $Na$, $Nb$, $Nc$, $Nd$ yield $[[a, b], [c, d]] \equiv 1$ mod $N$ and hence $G'' \leq N$.

**(Case 1)** We prove the result when $G$ has an infinite abelian normal subgroup $A$. So we can take $G = AF$.

Suppose $A$ has a non-trivial divisible subgroup. Since $G$ is periodic, we also may take $G = HF$ where $H = (C_{p^\infty})^n$ for some prime $p$, integer $n > 0$ and $n$ is minimal subject to $H \lhd G$ and assume that $1 \neq G'' \leq H$, $G''' = 1$. We will prove the case when $n$ is infinite at a later time.

For each $g \in G$, $g$ induces a homomorphism of $H$ to $[g, H]$ by $h \longmapsto [g, h]$ for $h \in H$, hence either $[g, H] = 1$ or $[g, H]$ is infinite. Therefore, if $[F', H]$ is finite, then $[F', H] = 1$. Easy computation shows that for any $g_i \in F$ and $h_i \in H$ $(i = 1, 2, 3, 4)$, $[[h_1 g_1, h_2 g_2], [h_3 g_3, h_4 g_4]] = 1$ implies $[[g_1, g_2], [g_3, g_4]] = 1$,

which contradicts $F \notin \mathfrak{A}^2$. Hence $[F', H]$ is infinite. By the minimality of $n$ and $[F', H] \lhd G = HF$, we have $G'' \leq H = [F', H] = [F', [F', H]] \leq G''$ and hence $[F', H] = H = G''$.

Pick $g_1, g_2$ in $F$ such that $[g_1, g_2]$ does not centralize $[g, H]$ for some $g \in F$. Let $X = \{x \in H \mid [[x, g], [g_1, g_2]] \neq 1\}$, then $X$ is infinite since $[F', H] = [F', [F', H]]$ is infinite. Therefore $[[h_1 g_1, h_2 g_2], [h_3 g, x]] = [[g_1, g_2], [g, x]] \neq 1$ for all $x \in X$, $h_i \in H$ which contradicts $G \in \mathfrak{A}_*^2$.

Suppose $A$ has no non-trivial quasicyclic $p$-group. If every $p$-component of $A$ is finite where $p$ is a prime, then the subgroup $B$ generated by all $p$-components of $A$ for $p$ which does not divide the order of $F$ is an infinite normal subgroup of $G$.

We may take $BF$ as a counter example, but $B \cap F = 1$ and $F \cong BF/B \in \mathfrak{A}^2$. Hence $A$ has an infinite $p$-component for some prime $p$ and hence has an infinite basic subgroup so we can assume $A$ is an elementary abelian $p$-group (the same argument can be applied for $H \cong (C_{p^\infty})^\infty$ ). Thus $A$ and $G$ are residually finite, the proof is complete in this case by (3.1)Lemma.

**(Case 2)** Suppose $G$ has no infinite abelian normal subgroups. We may take $G = HF$ where $G'' \leq H \in \mathfrak{A}^2$, $F \notin \mathfrak{A}^2$, both $H'$ and $F$ are finite, $H$ is infinite normal in $G$.

If $[F', H]$ is finite, then $(F')^G = (F')^{HF} = F'[F', H]$ is finite, the centralizer $C$ of $(F')^G$ in $H$ has finite index and hence it is infinite normal in $G$. Replacing $H$ by $C$, we may assume $[F', H] = 1$.

On the other hand if $[F', H]$ is infinite, then it is not abelian by the assumption and $[F', H] \lhd G$ is easily checked. Thus $HF' \notin \mathfrak{A}^2$ and we choose

a finite subgroup $F_1 \notin \mathfrak{A}^2$ of $HF'$ such that $HF_1 \notin \mathfrak{A}^2$ but $F_1' \leq H$ for $(HF')' \leq HF'' \leq H$. Hence $[F_1', H]$ is a finite subgroup of $H'$. As above, we can reduce to the case $[F_1', H] = 1$. Therefore, in either case, we may take $G = HF$ with $[F', H] = 1$. Furthermore $H/H'$ is infinite abelian and $G/H' \in \mathfrak{A}^2$ by (Case 1). So $G'' \leq H'$

Now we may assume $G''$ is minimal normal in $G$ and of exponent $p$. Let $C(H')$ be the centralizer of $H'$ in $H$, then $C(H') \triangleleft G$ and $G/C(H')$ is finite. Replacing $H$ by $C(H')$, we have $G'' \leq H' \leq Z(H)$ and hence $H$ is nilpotent. By the minimality of $G''$ and $F'' \triangleleft G$, we also may assume $F'' = G''$.

Write $K = G' \cap H$, then $|G' : K| < \infty$. Hence $K$ is infinite. Otherwise $G$ is an $FC$-group. Put $J = KF$. If $K' = 1$, then replacing $H$ by $K$, it is reduced to the (Case 1). We may assume $K' \neq 1$ and hence $K' = G''$ from $K' \triangleleft G$ and the minimality of $G''$. Also $G'' = F'' \leq J'' \leq G'' = K'$. Replacing $H$ by $K$ and hence $G$ by $J$, we may take $G'' = H'$. $G''$ still remains minimal.

For primes $p$, $q$, $p \neq q$, the $q$-component $H_q$ of $H$ is finite, otherwise consider $L = H_qF$ in which $L'' \cap H_q = F'' \cap H_q = 1$, which contradicts $1 \neq F'', F \notin \mathfrak{A}^2$ for $L/H_q \in \mathfrak{A}^2$ and $F'' \leq H_q$. Since $H$ is periodic nilpotent, $H$ can be chosen a $p$-group.

If $Z(H)$ is infinite, then it is reduced to the (Case 1). If not, then $H/Z(H)$ has a finite exponent, so does $H/H'$. Hence $H/H'$ contains a $G$-invariant infinite elementary $p$-subgroup $K/H'$. As before, we may assume $K' \neq 1$, and hence $K' = G'' = H'$. Replacing $H$ by $K$, we have $H' = G'' \leq Z(H) \leq H$, $H/H'$ is an infinite abelian elementary $p$-group and $H'$ is of exponent $p$.

Now let $A_1/H'$ be a minimal $G$-invariant subgroup of $H/H'$ and $C_1$ be $C_H(A_1)$. Then $C_1 \lhd G$ and $|G : C_1|$ is finite.

By the residual finiteness of $G/H'$, we can choose $H_1 \lhd G$ such that $G/H_1$ is finite and $A_1 \cap H_1 = H'$. Clearly we can take $H_1 \leq C_1$. Now let $A_2/H'$ be a minimal $G$-invariant subgroup of $H_1/H'$ and $C_2$ be $C_{H_1}(A_2)$.

Then $C_2 \lhd G$, $[A_1, A_2] = 1$ and $|G : C_2|$ is finite. Choose $H_2 \leq C_2$, $H_2 \lhd G$ such that $|G : H_2|$ is finite with $A_2 \cap H_2 = H'$. Continuing this process, we get a subgroup $N/H'$ of $H/H'$ which is the direct product of infinitely many minimal $G$-invariant finite subgroups $\bar{A}_i = A_i/H'$ where $[A_i, A_j] = 1$ for all $i \neq j$. Set

$$X_1 = \langle A_{4n+1} \mid n = 0, 1, 2, \ldots \rangle \qquad X_2 = \langle A_{4n+2} \mid n = 0, 1, 2, \ldots \rangle$$

$$X_3 = \langle A_{4n+3} \mid n = 0, 1, 2, \ldots \rangle \qquad X_4 = \langle A_{4n} \mid n = 0, 1, 2, \ldots \rangle.$$

Then each $X_i$ is infinite and normal in $G$. Pick $g_i \in G$ for $i = 1, 2, 3, 4$ such that $[[g_1, g_2], [g_3, g_4]] \neq 1$, then for each $x_i \in X_i$, $[[x_1 g_1, x_2 g_2], [x_3 g_3, x_4 g_4]] = [[g_1, g_2], [g_3, g_4]]$ since $[F', H] = 1$ and $[x_i, x_j] = 1$ for all $i \neq j$. This contradicts $G \in \mathfrak{A}^2_*$. ♠

**(3.4)Theorem.** *If $G \in \mathfrak{A}^2_*$ is locally soluble then $G$ is metabelian.*

**Proof.** Since $G/K \in \mathfrak{A}^2$ for any infinite normal subgroup $K$, every $G$-invariant proper subgroup of $G''$ is finite. If $G''$ is minimal normal in $G$, then $G''$ is abelian (See Theorem 5.5.1 [$RO_3$]). Therefore $G$ is soluble and $G \in \mathfrak{A}^2$ by the previous Theorem. Therefore we can choose $1 \neq A \lhd G$ and $A \lneq G''$. If $C = C_{G''}(A) \lneq G''$, then $C$ is finite, also $|G'' : C|$ and $G''$ are finite. Hence $G$ is soluble.

It follows that every proper $G$-invariant subgroup of $G''$ is contained in $Z(G'')$. Hence $G''/Z(G'')$ is minimal normal in $G/Z(G'')$ and so abelian. Therefore $G \in \mathfrak{A}^2$. ♠

**(3.5)Lemma.** *Suppose a periodic group $G \in \mathfrak{A}_*^2$ has an infinite abelian subgroup. If $F$ is any finite subgroup, then there exists an infinite metabelian subgroup containing $F$.*

**Proof.** Let $A$ be an infinite abelian subgroup and $x, y$ be in $F = G_0$. By considering $A, A, Ax, Ay$, we have a metabelian subgroup $\langle a_0, b_0, c_0x, d_0y\rangle$ for some $a_0$, $b_0$, $c_0$, $d_0 \in A$, so that $[[\langle a_0\rangle, x], [\langle a_0\rangle, y]] = 1$. Repeating this with $A\backslash\{a_0\}$ in place of $A$ and so on, there exists a cofinite subset $A_0$ of $A$ such that $[[\langle a\rangle, x], [\langle a\rangle, y]] = 1$ for all $a \in A_0$.

Since $G_0$ is finite, there exists a cofinite subset $A_1$ of $A_0$ such that $[[\langle a\rangle, x], [\langle a\rangle, y]] = 1$ for all $a$ in $A_1$ and all $x, y$ in $G_0$. For an element $1 \neq a_1 \in A_1$, let $N_1 = [\langle a_1\rangle, G_0]$. Then using the identities $[a_1, g]^{a_2} = [a_1a_2, g][a_2, g]^{-1}$ and $[a, g_1]^{g_2} = [a, g_2]^{-1}[a, g_1g_2]$ for any group element $a, g, a_i, g_i$, we can show $N_1 \lhd G_1 = \langle a_1, G_0\rangle$. But $G_1/N_1$ is finite and $N_1$ is abelian, hence $G_1$ and $N_1$ are finite.

Similiarly we construct $N_2 = [\langle a_2\rangle, G_1]$, $G_2 = \langle a_2, G_1\rangle$ where $a_2 \in A_2\backslash G_1$ for some cofinite subset $A_2$ of $A_1$ and $[[\langle a_2\rangle, g_1], [\langle a_2\rangle, g_2]] = 1$ for all $g_1$, $g_2 \in G_1$ and all $a_2 \in A_2$. Of course, $N_2$ is an abelian normal subgroup of $G_2$, both groups are finite. Continuing this process , we get an infinite strictly increasing tower of groups, $F = G_0 \subset G_1 \subset G_2 \subset \cdots$. Furthermore, $[G_{i-1}, \langle a_i\rangle] \equiv 1$ mod

$N_i$ for all $i \geq 1$. So we can write

$$G_1 = G_0\langle a_1\rangle N_1$$

$$G_2 = G_1\langle a_2\rangle N_2 = G_0\langle a_1\rangle N_1\langle a_2\rangle N_2$$

$$G_i = G_0\langle a_1\rangle N_1\langle a_2\rangle N_2 \cdots \langle a_i\rangle N_i$$

Let $K_n$ be the subgroup generated by the elements of $\{a_i, N_i \mid i = 1, 2, \ldots, n\}$ and $K = \cup_{n=1}^{\infty} K_n$. Clearly $K_1$ is soluble. By induction on $n$, we assume $K_{n-1} = \langle a_i, N_i \mid i \leq n-1\rangle$ is soluble. $N_n$ is abelian and normal in $K_n = \langle K_{n-1}, a_n, N_n\rangle$ and $\langle a_n\rangle$ is central in $K_n$ module $N_n$, which is therefore abelian-by-cyclic-by-soluble and hence soluble. It follows that $K$ is locally soluble and clearly infinite. As $F$ normalizes $K$, $FK/K$ is finite and hence metabelian. Therefore $FK$ is locally soluble and hence metabelian . ♠

From now on, for the further study of $\mathfrak{A}_*^2$-groups, we may assume that every finite subgroup is metabelian by (3.5)Lemma and Ramsey Theorem. Moreover we will show in (3.7) that it is enough to look at simple $\mathfrak{A}_*^2$-groups, in order to prove that $\mathfrak{A}_*^2$-groups are actually metabelian.

**(3.6)Theorem.** *If $G \in \mathfrak{A}_*^2$ is infinite locally finite then $G \in \mathfrak{A}^2$.*

**Proof.** It follows from (3.5)Lemma.

**(3.7)Reduction to Simplicity.** Suppose $G \in \mathfrak{A}_*^2$ but $G \notin \mathfrak{A}^2$. If $G^{(3)}$ is finite, then $G''$ is an $FC$-group where $G^{(3)} = [G'', G'']$. By (3.1)Lemma and (3.3)Theorem, we may assume $G''$ is finite and $G'$ is infinite.

Therefore $G'$ is an $FC$-group and hence $G' \in \mathfrak{A}^2$, it follows that $G$ is soluble and hence metabelian. If $G^{(3)}$ is infinite, then $G/G^{(3)} \in \mathfrak{A}^2$ which implies $G^{(3)} = G''$. Suppose $G''$ has an infinite proper normal subgroup $N$, then $G''/N \in \mathfrak{A}^2$, which implies $G^{(4)} \leq N$, contradicting to $G''$ perfect.

Let $N$ be a finite normal subgroup $N$ of $G''$, then $C_{G''}(N)$ has finite index in $G''$. Since $G''$ has no infinite normal subgroups, $C_{G''}(N) = G''$. Hence the centre $Z$ of $G''$ is the unique maximal normal subgroup of $G''$ so that $G''/Z$ is simple. By (3.6)Theorem, we may assume $G$ is a simple group .

**(3.8)Theorem.** *Any infinite group $G \in \mathfrak{A}_*^2$ is metabelian.*

**Proof.** By (3.7)Reduction, (3.6)Theorem and (3.2)Theorem, we may assume $G$ is finitely generated periodic simple. Suppose there exists a subgroup $C$ isomorphic to $C_{p^\infty}$ for some prime $p$, then for each pair $x, y \in G$, by considering sets $C$, $xC$, $C$, $yC$ we can say $\langle [\langle c \rangle, x], [\langle c \rangle, y] \rangle$ is abelian for all but finitely many $c$ in $C$. Hence for all $c \in C$, $\langle [\langle c \rangle, x], [\langle c \rangle, y] \rangle$ is abelian and therefore $[C, G]$. is abelian. Since $G$ is non-abelian, $[C, G] = 1$, therefore $G$ contains no $C_{p^\infty}$-subgroup.

Let $A$ be an infinite abelian subgroup and $S = S(x, y)$ denote the set of all $a \in A$ such that $\langle [\langle a \rangle, x], [\langle a \rangle, y] \rangle$ is abelian, then $S$ is a cofinite set of $A$ and $a \in S$ implies $\langle a \rangle \subseteq S$. If $A$ contains elements of infinitely many distinct prime orders, then choose a prime $p$ and an element $c$ of order $p$ such that no element of the finite set $B = A \backslash S$ has order a multiple of $p$ and such that for

all $b \in B$, $bc \notin B$. Since $bc \in S$ and $(bc)^p = b^p \in S$, we have $\langle b \rangle = \langle b^p \rangle$, $b \in S$ and hence $S = A$, which is true for all $x$, $y$ in $G$. So $[\langle c \rangle, G]$ is abelian normal in $G$.

Hence we may assume $A$ has elements of only finitely many prime orders and hence has finitely many $p$-component. Since $A$ contains no $C_{p^\infty}$-subgroup, $A$ contains an infinite elementary $p$-subgroup; $\langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \cdots$. There exists $a_i \in A$ such that for all $b \in B$, $ba_i \notin B$ and hence $b^p \in S$, it follows that $A^p \subseteq S = S(x, y)$ for all $x$, $y$ in $G$.

If $A^p \neq 1$, then $[\langle a' \rangle, G]$ is abelian normal in $G$ for $1 \neq a' \in A^p$. Therefore every infinite abelian subgroup has prime exponent. If every metabelian subgroup is abelian, then any infinite subset of $G$ contains 4 elements which generate a metabelian subgroup and hence it is abelian. By B.H. Neumann's Theorem [$NE_1$], $G$ is centre-by-finite and $G \in \mathfrak{A}^2$ by (3.1)Lemma. Hence there exists an infinite non-abelian metabelian subgroup $H$ by Ramsey's Theorem and (3.5)Lemma. Let $H_0 = \langle h_1, h_2 \rangle$ with $[h_1, h_2] \neq 1$ for some $h_1, h_2$ in $H$.

**(Case 1)** We prove the theorem when $G$, $H$ are groups stated above and additionally $G$ is a $p$-group.

If $H'$ is finite, then $H$ is an $FC$-group. If $H'$ is infinite, then $L = H'H_0$ is nilpotent ( G. Baumslag, see Lemma 6.34 [$RO_1$] ). Let $M$ be the least term of normal closure series of $H_0$ in $L$ subject to $M$ infinite. Since $H_0$ is subnormal in $L$, $H_0^M$ is finite. Hence $H_0$ has the infinite centralizer in $L$.

In either case, there is an infinite abelian subgroup $D$ centralizing $H_0 = \langle h_1, h_2 \rangle$. For any $x, y \in G$, there exist $d_i \in D$, $i = 1, 2, 3, 4$ such that $T = \langle d_1h_1, d_2h_2, d_3x, d_4y \rangle \in \mathfrak{A}^2$. So $\alpha = [h_1, h_2] \in T'$, $[\alpha, d_3x] = [\alpha, x] \in T'$ and

$[a,y] \in T'$, and hence $[[\langle a\rangle, x], [\langle a\rangle, y]] = 1$ for all $x, y \in G$. Therefore $[\langle a\rangle, G]$ is a normal abelian subgroup of $G$. This contradiction induces the other case.

**(Case 2)** When $G$ is not a $p$-group with above statement.

By (Case 1) and (3.5)Lemma, all of the Sylow $p$-subgroup of $G$ are metabelian. If $G$ has an infinite non-abelian Sylow $p$-subgroup of $G$, then the same argument as (Case 1) would yield an abelian normal subgroup of $G$. Hence we can assume every infinite Sylow $p$-subgroup of $G$ is abelian.

Since $G$ contains a non-abelian metabelian subgroup, it contains a finite such subgroup $F$. So $F'$ contains a nontrivial $q$-element $f$ for some prime $q$. Let $A$ be any infinite abelian group so that $A$ is a $p$-group for some $p$. As (3.5)Lemma we construct $H = H(F, A)$ which contains $F$. If $H'$ is finite, then $C_H(F)$ is infinite and hence it contains infinite abelian subgroup centralising $F$. Then similar to last part of (Case 1), $[\langle f\rangle, G]$ is a normal abelian subgroup of $G$. Thus $H'$ is infinite, abelian of prime exponent and contains $F'$, therefore $H'$ is a $q$-group.

It follows that we can choose $A$ as an infinite Sylow $q$-subgroup of $G$ not containing $f$ for $\cap_{g\in G} A^g \lhd G$ and hence $\cap_{g\in G} A^g = 1$. $F$ remains finite as before. Now $[\langle a\rangle, F]$ is abelian for all but finitely many $a \in A$. As (3.5)Lemma we can begin the construction of $H = H(F, A)$ with $N_1 = [\langle a\rangle, F]$ and go on to consider $G_1 = \langle F, a\rangle$ so that $H$ contains $G_1$. As we have seen, $H'$ is a $q$-group. This implies that $[\langle a\rangle, F]$ is an abelian $q$-group for all but finitely many $a \in A$. In particular, $[\langle a\rangle, \langle f\rangle]$ is a $q$-group. Since $\langle a, f\rangle / [\langle a\rangle, \langle f\rangle]$ is $q$-group, $\langle a, f\rangle$ is a $q$-group and is contained in the infinite $q$-group $H'\langle a, f\rangle$, which is abelian for all infinite Sylow subgroups are abelian. Thus $[a, f] = 1$ and $C_A(f)$

contains all but finitely many elements of $A$ and hence all of $A$. Thus $\langle A, f \rangle$ is a $q$-group, which contradicts the choice of $A$ and $f \notin A$.

## CHAPTER 4

## LOCALLY SOUBLE GROUPS WITH A METABELIAN LAW ON INFINITE SETS

Throughout this chapter, $G$ denotes an infinite group. We deal with a similiar type of problem as chapter 3 with weaker condition. By $G \in \mathfrak{A}_i^2$ We mean every four infinite sets $X_1, X_2, X_3, X_4$ of $G$ contain $x_i \in X_i$ $(i = 1, 2, 3, 4)$ such that $[[x_1, x_2], [x_3, x_4]] = 1$. Clearly $G \in \mathfrak{A}_*^2$ implies $G \in \mathfrak{A}_i^2$, but we don't know the other inclusion which may actually imply $G \in \mathfrak{A}^2$. Anyhow we prove that if $G \in \mathfrak{A}_i^2$ is locally soluble then $G \in \mathfrak{A}^2$.

Parts of proof mimic (3.2), (3.3)Theorem. We also depend on the properties of $FC$-groups to prove main theorems. Subgroups of this rather special type will play an important role in this chapter. These subgroups were introduced by B.H. Neumann [NE$_2$] and hence we use the term $N$-group ( See [TO]).

An $N$-group $G$ of cardinality $\aleph$ (infinite) is an $FC$-group generated by elements $x_i, y_i$, $i \in I$, where $|I| = \aleph$ and the generators satisfy the conditions

$$[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \text{ if } i \neq j, \quad [x_i, y_i] = z_i \neq 1.$$

These relations are not to be understood to form a system of defining relations of $G$, there will in general be further relations, not consequences of these relation. But they must not imply $z_i = 1$ for any $i$. In particular, if all $z_i$ are chosen distinct then such an $N$-group is called an $N_1$-group.

The following properties of an $N$-group $G$ are readily verified from the definition. The subgroups $Z_i = \langle x_i, y_i \rangle$ are normal in $G$ and they jointly generate $G$. The centralizer $C(Z_i)$ is generated by all the subgroups $Z_j$ with

$j \neq i$ together with the centre of $Z_i$, and the centre $Z$ of $G$ is generated by the centre of all $Z_i$.

We shall show that any $FC$-group $G$ contains an $N$-group if $G/Z(G)$ is infinite, thus obtaining the main results by considering these groups in more detail. To do this, we quote the following two well known Theorems.

**(4.1)Theorem** (Neumann [NE$_3$]). *If $G$ is an $FC$-group then $G'$ is periodic and hence the elements of finite order form a subgroup of $G$ containing $G'$.*

**(4.2)Theorem** (Černikov [CE]). *A group $G$ is a finitely generated $FC$-group if and only if it is isomorphic to a subgroup of the direct product of a finite group and a free abelian group of finite rank.*

**(4.3)Lemma**. *Let $U$ be a subgroup of finite index in an $FC$-group $G$.*

(1) *If $|U/Z(U)|$ is finite, then $|G/Z(G)|$ is finite*

(2) *If $U'$ is finite, then $G'$ is finite*

**Proof.** There is a finitely generated normal subgroup $F$ of $G$ such that $FU = G$.

Since $|G/C_G(F)|$ is finite and $|G : Z(U)|$ is finite, we have $|G : C_G(F) \cap Z(U)|$ is finite, but $C_G(F) \cap Z(U)$ centralizes $FU = G$ and so $G/Z(G)$ is finite. In (2), we have $G' \leq U'(F \cap G')$. But $F \cap G'$ is contained in the periodic subgroup of $F$ by (4.1)Theorem and so is contained in a finite subgroup of $F$ by (4.2)Theorem. Therefore $G'$ is finite. ♠

**(4.4)Theorem.** *Let $G$ be an $FC$-group.*

(1) *If $|G/Z(G)|$ is infinite, then $G$ contains an infinite $N$-subgroup*

(2) *If $G'$ is infinite then $G$ contains an infinite $N_1$-group*

**Proof.** We define the generators $x_n$, $y_n$ $(n = 1,2,3,\dots)$ inductively. Suppose that we have defined $x_1, y_1, \dots, x_{n-1}, y_{n-1}$ and let $C = C_G(x_1, y_1, \dots, x_{n-1}, y_{n-1})$ so that $|G : C|$ is finite. By (4.3)Lemma, $C/Z(C)$ is infinite and in part (2), $C'$ is infinite.

Therefore $C$ contains elements $x_n, y_n$ such that $[x_n, y_n] \neq 1$, and part (2), we can choose $x_n, y_n$ so that $[x_n, y_n] \notin \{1, z_1, \dots, z_{n-1}\}$. ♠

Same results in chapter 3 hold for $\mathfrak{A}_1^2$ in place of $\mathfrak{A}_*^2$. Proofs require no change. Theses are listed below.

**(4.5)Lemma.** *$G \in \mathfrak{A}_1^2$ is metabelian if*

(1) *$G$ has the infinite centre*

(2) *$G$ is an $FC$-group*

(3) *$G$ is residually finite*

**Proof.** See (3.1)Lemma.

**(4.6)Lemma.** *$G \in \mathfrak{A}_1^2$ is metabelian if $C_G(G')$ has $G$-invariant infinite abelian subgroup.*

**Proof.** Let $C$ be a such subgroup of $G$ and $a, b, c, d$ in $G$. Consider four sets $aC, bC, cC, dC$. By $G \in \mathfrak{A}_1^2$, there exist $z_i \in C$ for $i = 1,2,3,4$ such that $[[az_1, bz_2], [cz_3, dz_4]] = 1$. This implies $[[a, b], [c, d]] = 1$. Hence $G \in \mathfrak{A}^2$. ♠

**(4.7)Corollary.** *$G$ is metabelian if $Z(G')$ is infinite.*

**(4.8)Lemma.** *If $G \in \mathfrak{A}_1^2$, then $[G', [G, N]\,] = 1$ for any infinite normal abelian subgroup $N$ of $G$.*

**Proof.** For any $a, b, c$ in $G$, consider sets $Na, Nb, Nc, N$ and let $S = S(a, b, c) \subset N$ be such that for any $s \in S$ there exist $n_i \in N$ $(i = 1,2,3)$ with

$[[n_1a, n_2b], [n_3c, s]] = 1$, so that $[[a, b]n, [c, s]] = [[a, b], [c, s]] = 1$ for some $n \in N$. Clearly, $N \backslash S$ is finite, $N = \langle S \rangle$. Since $[c, s_1 s_2] = [c, s_2][c, s_1]^{s_2} = [c, s_1][c, s_2]$ and $1 = [c, s_1 s_1^{-1}] = [c, s_1^{-1}][c, s_1]$ for all $s_1, s_2 \in S$, $[a, b]$ commutes with $[c, n]$ for all $n \in N$. Hence $[[a, b], [c, N]] = 1$, as desired.

**(4.9)Corollary.** *If $G \in \mathfrak{A}_1^2$ and $[G, N]$ is infinite for some infinite normal abelian subgroup $N$ of $G$, then $G \in \mathfrak{A}^2$.*

**Proof.** Since $[G, N] \leq G'$ and $[G, N] \leq Z(G')$, $G \in \mathfrak{A}^2$ by (4.7)Corollary.

**(4.10)Theorem.** *If $G \in \mathfrak{A}_1^2$ is a finitely generated soluble infinite group, then $G$ is metabelian.*

**Proof.** We prove by induction on the solubility length of $G$. We consider two cases;

**(Case 1)** We prove the result when $G$ has an infinite abelian normal subgroup $N$.

By the previous Lemma, we may assume $[N, G]$ is finite. Let $D = C_G[N, G]$. Clearly $[N, D, D] \leq [\,[N, G], D] = 1$, hence $[D', N] = 1$ by the Three Subgroups Lemma (See [RO$_3$]). Therefore, $D$ and $G$ are residually finite, so desired.

**(Case 2)** When $G$ has no infinite abelian normal subgroup.

By induction hypothesis, $G''$ is finite abelian and hence $G'$ is an $FC$-group. Since the centralizer $H$ of $G''$ in $G$ has finite index, it is enough to show $H \in \mathfrak{A}^2$ for being residually finite-by-finite is equivalent to residually finite. Replacing $G$ by $H$, we may assume $[G, G''] = 1$. Since $G$ is finitely generated, choose $\{t_1, \ldots, t_n\}$ as a set of generators of $G/G'$.

Let $t$ be any $t_i$ and $T$ be the subgroup generated by $[G',t]$ and $G''$. First we show $T = [G',\langle t\rangle]\ G'' \lhd G$ and each element of $T$ is of the form $[a,t]g''$ for some $a \in G'$, $g'' \in G''$. The following can be easily checked by using commutator's identities $[xy,z] = [x,z]^y[y,z]$ and $[x,yz] = [x,z][x,y]^z$ for any group elements $x, y, z$.

(1) $[a,t][b,t] = [ab,t]g''$ for some $g'' \in G''$, depending on choice of $a, b \in G'$

(2) $[a,t]^{-1} \in [a^{-1},t]G''$ for $[a,t][a^{-1},t] \in G''$ from (1)

(3) $[a,t^2] = [a,t][b,t] = [ab,t]g''$ for some $g'' \in G''$, $b = a^t$

(4) $[a,tt^{-1}] = 1$ implies $[a,t^{-1}] = [b,t]^{-1}$ where $b = a^{t^{-1}}$

(5) $[a,t][t,b] = [t,ba^{-1}]^a = [ba^{-1},t]^{-1}g''$ for some $g'' \in G''$

(6) $[a,t][t,b] \equiv [t,b][a,t] \mod G''$

(7) $[a,t^g] = [a,t[t,g]\,] = [a,t]g''$ for some $g'' \in G''$

where $a, b \in G'$, $g \in G$. (1)–(6) tell us the typical form of elements of $T$ and (7) implies $T \lhd G$. If $T_i = [G',t_i]G''$ is finite for all $i$, then $[G',G] \leq T_1T_2\cdots T_n$. Hence $G$ is finite-by-nilpotent and hence nilpotent-by-finite. (See Theorem 4.25 [$RO_1$]). Therefore $G$ is residually finite ( See P. Hall [PH]) and hence $G \in \mathfrak{A}^2$.

Now we may assume $T = [G',t]G''$ is infinite. By (Case 1), $T$ is non-abelian and has the finite centre. Recall $T \leq G'$ is an $FC$-group and $G''$ is central. By (4.4)Lemma, $T$ has an $N$-group generated by $x_i, y_i$ $(i \in I)$ subject to $[x_i\ y_i] \neq 1$, but $[x_i,y_j] = [x_i,x_j] = [y_i,y_j] = 1$ for all $i \neq j$. Let

$$x_i = [a_i,t]g_i'' \qquad y_j = [b_j,t]f_j''$$

for some $a_i, b_j \in G'$, $g_i'', f_j'' \in G''$. Then clearly for each $i \in I$,

$$[x_i, y_i] \neq 1 \quad \text{implies} \quad [[a_i, t], [b_i, t]] \neq 1,$$

$$[x_i, y_j] = 1 \quad \text{implies} \quad [[a_i, t], [b_j, t]] = 1 \text{ and etc.}$$

Let $i_0 \in I$ and $I$ be the disjoint union of $\{i_0\}$, $I_1$ and $J_1$ with $|I_1| = |J_1| = \infty$. Set

$$X_1 = \{a_{i_0} b_{i_1} \mid i_1 \in I_1\} \qquad X_2 = \{t a_{i_1} \mid i_1 \in I_1\}$$

$$X_3 = \{b_{i_0} a_{j_1} \mid j_1 \in J_1\} \qquad X_4 = \{t b_{j_1} \mid j_1 \in J_1\}.$$

Note that $a_i's$ $(b_i's)$ are not necessarily distinct, but $G''$ is finite, so $X_i$ is infinite. The set of commutators of elements of $X_1$ and $X_2$, denoted by $[X_1, X_2]$, consists of $[a_{i_0} b_{i_1}, t a_{i_1}] = [a_{i_0}, t][b_{i_1}, t] g''$, and $[X_3, X_4]$ consists of $[b_{i_0}, t][a_{j_1}, t] f''$, for some $g''$, $f'' \in G''$, depending choice of elements of $X_i$. But

$$\begin{aligned}[[a_{i_0}, t][b_{i_1}, t] g'', \ [b_{i_0}, t][a_{j_1}, t] f''] &= [[a_{i_0}, t][b_{i_1}, t], \ [b_{i_0}, t][a_{j_1}, t]] \\ &= [[a_{i_0}, t][b_{i_1}, t], \ [b_{i_0}, t]]^{g'} \text{ for some } g' \in G' \\ &= [[a_{i_0}, t], [b_{i_0}, t]]^{g} \text{ for some } g \in G\end{aligned}$$

Therefore $[[x_1, x_2], [x_3, x_4]] \neq 1$ for all $x_i \in X_i$, contradicting to $G \in \mathfrak{A}_1^2$. ♠

**(4.11)Theorem.** *If $G \in \mathfrak{A}_1^2$ is an infinite periodic soluble group then $G \in \mathfrak{A}^2$.*

**Proof.** We use induction on the solubility length of $G$. We may assume $G'$ is metabelian. Also $G'$ is infinite for otherwise $G$ is an $FC$-group.

**(Case 1)** We prove the result when $G$ has an infinite normal abelian subgroup.

Suppose $G$ is a counter example, then there exists a finite subgroup $F \notin \mathfrak{A}^2$. Hence we can take $G = FA$ where $A$ is infinite abelian normal in $G$. Suppose

$A$ has a divisible subgroup and hence a divisible $p$-subgroup $H$. Since $H$ is normal in $G$, $[G', [G, H]] = 1$ by (4.8)Lemma. If $[G, H]$ is infinite then $G \in \mathfrak{A}^2$.

If $[G, H]$ is finite then $[g, H]$ is finite for all $g \in G$. Now for each $g \in G$, $g$ induces a homomorphism of $H$ by $h \longmapsto [g, H]$. If $[g, H]$ is finite, then $[g, H] = 1$ and hence $[G, H] = 1$. So done by (4.5)Lemma when $[G, H]$ is finite. Therefore we may assume $A$ has no divisible subgroup.

If $A$ has finite $p$-components of $A$ for infinitely many primes $p$, then the subgroup $B$ generated by infinitely many finite $p$-components of $A$ for prime $p$ which does not divide the order of $F$ is normal in $G$. So we may take $G = BF$ as a counter example with $B \cap F = 1$. Since $B$ is infinite, $G/B \in \mathfrak{A}^2$ and hence $BF/B \cong F/B \cap F \cong F \in \mathfrak{A}^2$, contradicting to the choice of $F$.

It follows that $A$ is an infinite $p$-group and reduced. Hence $A$ has an infinite basic subgroup. (See [FU]) We may assume $A$ is an elementary abelian $p$-group, therefore $A$ is residually finite and hence $G$. So the result follows.

**(Case 2).** We prove the result of the other case.

We may assume $G$ has no infinite normal abelian subgroup and $G''$ is finite. We may take $G = AF$ where $F \notin \mathfrak{A}^2$ is finite, $A \in \mathfrak{A}^2$, $A'$ is finite abelian normal and $G' \leq A$.

Let $D = C_A(A')$ then $G/D$ is finite and $D \lhd G$ for $D = A \cap C_G(A')$. For $g \in G$, let $N(g)$ be the subgroup generated by $[g, D]A'$, then clearly

$$[g, D]^h (A')^h = [g^h, D]A' = [g, D][\,[g, h], D]A' = [g, D]A'$$

for all $h \in G$ and hence $N(g) \lhd G$. Since $A'$ is finite $N(g) \subset A$ is an $FC$-group. For $d_1, d_2$ in $D$, we have

$$[g, d_1 d_2] = [g, d_2][g, d_1]^{d_2} = [g, d_2][g, d_1]a'$$

for some $a' \in A'$. Hence $[g,d_1][g,d_2] \in [g,d_1d_2]A'$. Since $G$ is periodic $[g,d_1]^{-1} = [g,d_1]^\ell \in [g,d]A'$ for some $d \in D$ and $\ell > 1$. Therefore every element of $N(g)$ is of the form $[g,d]a'$ for some $d \in D$ and $a' \in A'$.

We choose a transversal $\{g_1, \cdots, g_n\}$ of $D$ in $G$ then $[G,D]A' = [g_1,D]\cdots[g_m,D]A'$. If $N(g)$ is finite for all $g \in G$ then $[G,D]$ and $\{g^D\}$ are finite for all $g \in G$, where $\{g^D\}$ denotes the conjugate class of $g$ in $D$. Hence $G$ is an $FC$-group.

Therefore, we may assume $N(g)$ is infinite for a fixed element $g \in G$. By (Case 1), the centre $Z$ of $N(g)$ is finite. Hence an $FC$-group $N(g)$ has an infinite $N$-subgroup generated by $x_i, y_i$ $(i \in I)$ for an infinite $I$, subject to $[x_i,y_i] \neq 1$, $[x_i,y_j] = [x_i,x_j] = [y_i,y_j] = 1$ if $i \neq j$. Set

$$x_i = [g,c_i]a_i' \qquad y_i = [g,d_i]b_i'$$

for $c_i, d_i \in D$ and $a_i', b_i' \in A'$. This setting is possible for every element of $N(g)$ is of this type.

Recall $A'$ is finite abelian normal, $D \leq A$, $[D,A'] = 1$ and hence $[\,[g,D],A'] = 1$. Choose $i_0 \in I$ and let $I$ be the disjoint union of $I_1$, $J_1$ and $\{i_0\}$ where $|I_1| = |J_1| = \infty$. Let

$$X_1 = X_3 = \{gd \mid d \in D\}$$

$$X_2 = \{c_{i_0}d_{i_1} \mid i_1 \in I_1\}$$

$$X_4 = \{d_{i_0}d_{j_1} \mid j_1 \in J_1\}$$

then, $[X_1,X_2]$ consists of $[gd, c_{i_0}d_{i_1}] = [g,d_{i_1}][g,c_{i_0}]a'$ for some $a' \in A'$, and $[X_3,X_4]$ consists of $[gd, d_{i_0}d_{j_1}] = [g,d_{j_1}][g,d_{i_0}]b'$ for some $b' \in A'$. Hence for $x_i \in X_i$, $[[x_1,x_2],[x_3,x_4]] = [[g,c_{i_0}],[g,d_{i_0}]]^g$ for some $g \in G$, which is not equal to 1, contradicting to $G \in \mathfrak{A}_1^2$. ♠

**(4.12)Theorem.** *If* $G \in \mathfrak{A}_1^2$ *is locally soluble then* $G$ *is metabelian.*

**Proof.** Since $G/K \in \mathfrak{A}^2$ for any infinite normal subgroup $K$, every $G$-invariant proper subgroup of $G''$ is finite. If $G''$ is minimal normal in $G$, then $G''$ is abelian (See Theorem 5.5.1 [$RO_3$]). Therefore $G$ is soluble and $G \in \mathfrak{A}^2$ by the previous Theorem. Therefore we may assume $1 \neq A \lhd G$ and $A \lneq G''$. If $C = C_{G''}(A) \lneq G''$, then $C$ is finite, $|G'' : C|$ and $G''$ are finite and hence $G$ is soluble.

It follows that every proper $G$-invariant subgroup of $G''$ is contained in $Z(G'')$. Hence $G''/Z(G'')$ is minimal normal in $G/Z(G'')$ and hence abelian. Therefore $G \in \mathfrak{A}^2$. ♠

(4.12)Theorem tells us it is possible to assume that $G$ is simple for further study of $\mathfrak{A}_1^2$-groups as we have seen in (3.7)Reduction to Simplicity.

## CHAPTER 5

## EXTENSIONS OF A PROBLEM OF PAUL ERDÖS ON QUASI-HAMILTONIAN GROUPS

In this chapter, $G$ denotes an infinite group. A non-abelian group $G$ is said to be *Hamiltonian* if every subgroup is normal, and to be *quasi-Hamiltonian* if any two subgroups permute as subgroups.

The structure of Hamiltonian groups is well known (See Dedekind and Baer 5.3.7 [$RO_3$]). Quasi-Hamiltonian groups have been studied by K. Iwasawa in [$IW_1$], [$IW_2$]. Joining quasi-Hamiltonian groups with a variation of Erdös problem, P. Longobardi, M. Maj, A.H. Rhemtulla and H. Smith [LMRS] have proved the following, which extends the result of [CLRW].

**(5.1)Theorem** (LMRS). *Suppose every infinite set of subgroups of $G$ contains a pair that permute. Then a finitely generated $G$ is centre-by-finite, conversely a finitely generated centre-by-finite group satisfies the hypothesis.*

Above is not true in general and a counter example was provided in [CLRW] which is based on a group constructed by Iwasawa [$IW_2$] and Napolitani [NA].

We are concerned here about finitely generated groups satisfying (5.2).

**(5.2)** any two infinte sets $\mathbf{X}, \mathbf{Y}$ of subgroups of $G$ contain $H \in \mathbf{X}$, $K \in \mathbf{Y}$ such that $HK = KH$.

**(5.3)Theorem.** *Any finitely generated group $G$ is quasi-Hamiltonian if and only if $G$ satisfies (5.2).*

**Proof.** Clearly it is enough to show $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$ for all $a, b \in G$.

(5.2) is a stronger condition than that of (5.1)Theorem and hence $G$ has a torsion-free element $z \in Z(G)$.

**(Case 1)** Suppose $a, b$ are elements of finite order.

Let $(\langle a\rangle\langle b\rangle)^2$ be the set of elements of the form $a^{\ell_1}b^{\ell_2}a^{\ell_3}b^{\ell_4}$ for $\ell_i \in \mathbf{Z}$. Then this set is finite and hence we can choose $0 \neq m \in \mathbf{Z}$ such that $\langle z^m\rangle \cap (\langle a\rangle\langle b\rangle)^2 = 1$. Let

$$X = \{\ \langle a\rangle\langle z^m\rangle,\ \langle a\rangle\langle z^{2m}\rangle, \dots\}$$

$$Y = \{\ \langle b\rangle\langle z^m\rangle,\ \langle b\rangle\langle z^{2m}\rangle, \dots\}$$

By the hypothesis, $\langle a\rangle\langle b\rangle\langle z^{km}\rangle = \langle b\rangle\langle a\rangle\langle z^{km}\rangle$ for some $k \in \mathbf{Z}$, and hence $\langle a\rangle\langle b\rangle \subset \langle b\rangle\langle a\rangle\langle z^{km}\rangle$. Therefore, $\langle a\rangle\langle b\rangle \subset \langle b\rangle\langle a\rangle$ and similiarly we have $\langle b\rangle\langle a\rangle \subset \langle a\rangle\langle b\rangle$.

**(Case 2)** Suppose $a, b$ are elements of infinite order.

Since $G$ is centre-by-finite, there exists $n$ such that $a^n$, $b^n$ are central in $G$. Set

$$X = \{\ \langle a\rangle,\ \langle a^{n+1}\rangle,\ \langle a^{2n+1}\rangle, \dots\}$$

$$Y = \{\ \langle b\rangle,\ \langle b^{n+1}\rangle,\ \langle b^{2n+1}\rangle, \dots\}$$

then, $\langle a^{tn+1}\rangle\langle b^{sn+1}\rangle = \langle b^{sn+1}\rangle\langle a^{tn+1}\rangle \subset \langle b\rangle\langle a\rangle$ for some $t$ and $s$. For all $i$, $j$, $a^ib^ja^{tni}b^{snj} \subset \langle b\rangle\langle a\rangle$ and hence $\langle a\rangle\langle b\rangle \subset \langle b\rangle\langle a\rangle$. Similiarly we have $\langle b\rangle\langle a\rangle \subset \langle a\rangle\langle b\rangle$.

**(Case 3)** Suppose the order of $a$ is infinite and the order of $b$ is finite.

Choose $0 < n \in \mathbf{Z}$ such that $a^n \in Z(G)$ and $\langle a^n \rangle \cap \langle b \rangle = 1$. Set

$$X = \{ \langle a \rangle, \langle a^{n+1} \rangle, \langle a^{2n+1} \rangle, \dots \}$$

$$Y = \{ \langle b \rangle, \langle b \rangle\langle a^n \rangle, \langle b \rangle\langle a^{2n} \rangle, \dots \}$$

Then, $X$ and $Y$ are infinite. Hence $\langle a^{tn+1} \rangle\langle b \rangle\langle a^{sn} \rangle = \langle b \rangle\langle a^{sn} \rangle\langle a^{tn+1} \rangle$ for some $s, t$ Therefore, $\langle a \rangle\langle b \rangle = \langle b \rangle\langle a \rangle$, and the result follows. ♠

**(5.4)Corollary.** *Any group $G$ having an element of infinite order is quasi-Hamiltonian if and only if $G$ satisfies (5.2).*

**Proof.** Let $x$ be an element of infinite order. Then a finitely generated infinite group $\langle x, a, b \rangle$ is quasi-Hamiltonian for each choice of $a, b$. Hence $\langle a \rangle\langle b \rangle \subset \langle b \rangle\langle a \rangle$. ♠

Suppose $G$ is an infinite group satisfying (5.2). If $G$ is not periodic, then by (5.4)Corollary $\langle a, b \rangle = \langle a \rangle\langle b \rangle$ for each $a, b \in G$. It is shown in [LMRS] that a periodic group satisfying the hypothesis (5.1)Theorem is locally finite. Hence periodic groups satisfying (5.2) are locally finite. So we have the following;

**(5.5)Corollary.** *Suppose $G$ satisfies (5.2). The set of all elements of finite order in $G$ forms a characteristic subgroup*

(5.3)Theorem is not true in general. Clearly a counter example must be locally finite.

**(5.6)Example.** Let $D_8$ be the dihedral group generated by $a, b$ with relations $a^4 = b^2 = 1$. It is easy to see that $\langle a^2 \rangle = D_8' = Z(D_8)$ and $D_8$ is not quasi-Hamiltonian; for example, $\langle ab \rangle\langle a^2 b \rangle \neq \langle a^2 b \rangle\langle ab \rangle$.

Let $\frac{1}{2}$ be the element of order 2 in the quasi-cyclic 2-group $C_{2^\infty}$, then $N = \langle (a^2, \frac{1}{2}) \rangle$ is normal in $D_8 \oplus C_{2^\infty}$. Let $G$ be the quotient of $D_8 \oplus C_{2^\infty}$ by $N$. Let $\mathbf{X}$, $\mathbf{Y}$ be two infinite sets of subgroups of $G$. Since any subgroup of $C_{2^\infty}$ is considered normal in $D_8 \oplus C_{2^\infty}$, we may assume no subgroups in $\mathbf{X}$, $\mathbf{Y}$ is a subgroup of $C_{2^\infty}/N$ (this notation makes sense since $a^2$ is identified with $\frac{1}{2}$).

Therefore, we can choose $H \in \mathbf{X}$, $K \in \mathbf{Y}$ such that $\overline{x_1 + y_1} \in H$, $\overline{x_2 + y_2} \in K$ with $|y_1| > 4$, $|y_2| > 4$ for $x_i \in D_8, y_i \in C_{2^\infty}$. Then $x_1^4 + y_1^4 \in H$ implies $a^2 \in H$. Similiarly $a^2 \in K$. Now pick any nontrivial elements $d_1 + c_1 \in H$, $d_2 + c_2 \in K$ with $d_i \neq 1$ in $D_8$, then

$$\begin{aligned}(d_1 + c_1)(d_2 + c_2) &= d_1 d_2 + c_1 c_2 \\ &= d_2 d_1 [d_1, d_2] + c_2 c_1 \in KH\end{aligned}$$

This implies $HK \subset KH$. Similiarly $KH \subset HK$. On the other hand, $G$ contains a subgroup isomorphic to $D_8$. Hence $G$ is not quasi-Hamiltonian but does satisfy (5.2).

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